A novel varistructure grey forecasting model with speed adaptation and its application

Shoujun Li[a,b,*], Xiaoping Ma[a], Muhammad Ikram[c]

[a] School of Information and Control Engineering, China University of Mining and Technology, Xuzhou 221116, P.R.China
[b] School of Mechanical and Electrical Engineering, Suqian College, Suqian 223800, P.R.China
[c] College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, PR China

Abstract

Traditional grey forecasting models can not achieve satisfactory predictive accuracy on approximate non-homogeneous exponential sequence with acceleration, velocity and constant perturbation terms. Hence, it is hard to forecast the developing trend of non-homogeneous exponential data sequences widespread in the real world. It is for this reason that an extended intelligent grey forecasting model with variable speed and adaptive structure(VSSGM for short) is provided in this paper. The proposed model possesses three advantages, namely, strong adaptability to perturbation sequence, self-changing structure, and outstanding accuracy in simulation and prediction. One prominent property is that it can simulate/forecast any given inhomogeneous sequence with acceleration, velocity and constant disturbances, which are the most common perturbations widespread in nature and society. And the proposed VSSGM model can be reducible to DGM(1,1), NDGM, SAIGM and VCGM models in theory. Therefore, the proposed model outperforms the traditional grey models. In order to verify the effectiveness, universality, practicality and feasibility of the model, we conducted an experiment to analyze the precision of the model and employ it to simulate China's industrial added value from 2006 to 2015, then utilize it to forecast the industrial added values from 2017 to 2021. Comparing with the real statistic data, it shows that the proposed model has better prediction performance than traditional grey models.

Keywords:
Grey forecasting model, Self changing structure, Speed adaption, Variable-speed perturbation, China's industrial added value

1. Introduction

The nature and society are practically a generalized energy system[3], accumulation or release of the energy in which follows quasi-exponential variation trend[21]. In order to forecast the change rule of the system, the grey prediction theory was born and flourished[9]. As the basis of this theory, GM(1,1) model $x^{(0)}(k) + az^{(1)}(k) = b$ has achieved high prediction accuracy with its forecasting sequence obeying homogeneous exponential law as follows[4].

$$\hat{x}^{(0)}(k+1) = (1 - \mathrm{e}^{a})\left(x^{(0)}(1) - \frac{b}{a}\right)\mathrm{e}^{-ak}, k = 1, 2, \ldots, n. \tag{1}$$

Therefore, the predictive sequence given by Eq.(1) obeys pure homogeneous exponential law which reflects the accumulating or releasing of energy in a general energy system to be considered. It has a fairly high prediction accuracy and has been widely applied in many domains in nature and society, such as industry [26], agriculture [22], economics [26] and energy [24].

*Corresponding author

Email addresses: lishoujunbox@126.com (Shoujun Li), xpma@cumt.edu.cn (Xiaoping Ma), mikram@nuaa.edu.cn (Muhammad Ikram)

However, there does not exist ideal energy system[8]. As a matter of fact, accumulation of energy usually runs through three stages, i.e., incubation stage, uniform motion and acceleration phase to a new equilibrium state. Therefore, absolute homogeneous exponential trend does not exist at all, instead, practical variation trend of parameter in real system conforms to non-homogeneous exponential law. For example, there are three possible variation trends as shown in Fig. 1, standing for approximate exponential sequences with perturbations of constant, velocity and acceleration terms respectively. This means the simulation error generated by GM(1,1) model is practically inevitable. In order to improve the accuracy of the GM(1,1) model, scholars have made many studies of entire modeling process in the past 30 years. These studies mainly focus on the following five aspects, namely $X^{(0)}, X^{(1)}, Z^{(1)}, x^{(0)}(1), \varepsilon$ [23].

In order to get a quasi-homoeneous sequence, it needs to improve the smoothness of raw data sequence $X^{(0)}$ through some data processing methods, such as function transformation [11], buffering algorithm [14] and Fourier transform [5]. $X^{(1)}$ is an accumulated generating sequence of $X^{(0)}$, and it is often used for whitening a grey process [17]. $Z^{(1)}$ is the background value of GM(1,1) model, and its general formula is $z^{(1)}(k) = \alpha x^{(1)}(k) + (1-\alpha)x^{(1)}(k-1)$ [7], the generation coefficient α in this formula is often optimized by employing some intelligent algorithms such as particle swarm algorithm [16] and artificial bee colony optimization method [12]. It can be seen from Eq.(1) that the forecast sequence of GM(1,1) is related to iterative datum $x^{(0)}(1)$. When simulated error ε is bigger than expected, residual correction model [15] can be adopted to improve the precision.

No matter what optimization method is used, GM(1,1) model cannot simulate or forecast exactly any non-homogeneous exponential sequence for reasons of existence of structural contradiction between the model and the sequence to be predicted. Consequently, many GM(1,1) derived models have been brought forward gradually, mainly concentrated on structural optimization. There are 6 derived models with the improvement in structures different from that of GM(1,1) model, including DGM(1,1) model, NGM(1,1,k) model, NDGM model, SAIGM model, GM(1,1,t^{α}) model and grey linear regression model.

As is known that the estimated parameter vector $\hat{a} = (a, b)^{\mathrm{T}}$ comes from GM(1,1) model itself through ordinary least square method, but its time response sequence $\hat{x}^{(1)}(k)$ and regressive reduction sequence $\hat{x}^{(0)}(k)$ are worked out by continuous whitenization equation. This implies that simulative error is inevitable and GM(1,1) model cannot simulate pure homogeneous exponential sequence without error. For this reason, Literature [19] proposed DGM(1,1) model with the formal form as follows.

$$x^{(1)}(k+1) = \beta_1 x^{(1)}(k) + \beta_2, \tag{2}$$

which can simulate homogeneous exponential sequence $x^{(0)}(k) = a_0 p^k$ without error, as shown in Table 1. Moreover, 3 years later, paper [20] put forth NDGM model with the structure

$$\begin{cases} x^{(1)}(k+1) = \beta_1 x^{(1)}(k) + \beta_2 k + \beta_3, \\ x^{(1)}(1) = x^{(0)}(1) + \beta_4. \end{cases} \tag{3}$$

which can very precisely simulate and forecast non-homogeneous exponential sequence $x^{(0)}(k) = a_0 p^k + a_1$ with a constant term a_1, as shown in Table 1. In other way, paper [1] proposed NGM(1,1,k) model with the basic form

$$x^{(0)}(k) + a z^{(1)}(k) = bk. \tag{4}$$

It can but simulate inhomogeneous exponential sequence approximately, as shown in Table 1. What is more, its structure is not reasonable if no constant term attached to the right-hand side of the Eq.(4) [25]. After attachment of a constant term c, SAIGM model[23] will be obtained

$$x^{(0)}(k) + a z^{(1)}(k) = bk + c, \tag{5}$$

which can simulate and forecast non-homogeneous exponential sequence $x^{(0)}(k) = a_0 p^k + a_1$ without bias, as shown in Table 1. Although grey linear regression model [10] $\hat{x}^{(1)} = c_1 e^{-ak} + c_2 k + c_3$ has been widely used, it can only simulate approximately the inhomogeneous sequence, and its development index $-a$ and c_1, c_2, c_3 are not solved at the same time but solved in two steps respectively [18], consequently, resulting in lower forecast precision than expected. The GM(1,1,t^{α}) model proposed by literature [13] has stronger adaptability and can track different variation tendency of data sequence, but it can simulate sequences with special laws. For example, when $\alpha = 2$, the model can only simulate the following sequence with high accuracy.

$$\hat{x}^{(0)}(k) = f_1(a, b, c, x^{(0)}(1))\mathrm{e}^{-ak} + f_2(a, b)k + f_3(a, b). \tag{6}$$

Where, f_1, f_2 and f_3 are all constrained functions whose values are determined by a, b, c and iterative datum $x^{(0)}$. This means that GM(1,1,t^α) is not suitable to any given data sequence such as $x^{(0)}(k) = a_0 p^k + a_2 k + a_1$ shown in Table 1.

How well the computing method is selected has significant influence on the simulation accuracy. There are two methods employed usually, one is whitenization method, another is connotation method[6]. Whitenization method means that the regressive reduction formula $\hat{x}^{(0)}(k)$ is obtained by solving the whitenization equation. Connotation method means $\hat{x}^{(0)}(k)$ is obtained by solving the difference equation of the model directly. For example, whitenization method has been used in GM(1,1), NGM(1,1,k), GM(1,1,t^α) models, while connotation method has been used in DGM(1,1), NDGM, SAIGM models. In theory, the latter has higher accuracy than the former, in that the latter is a direct computing method without any form transformation.

Table 1
Comparisons in structural adaptability, computing method, sequence tracking ability and forecasting accuracy for GM(1,1) model and its derived models.

No.	Grey forecasting models	Adaptability of structure				Computing method		Sequence tracking ability	accuracy
		AS_1	AS_2	AS_3	AS_4	$Method_1$	$Method_2$		
1	GM(1,1)	✓				✓		exp	with error
2	DGM(1,1)	✓					✓	exp + const	without error
3	NGM(1,1,k)	✓	✓			✓		exp + const	with error
4	NDGM	✓	✓				✓	exp + const	without error
5	SAIGM	✓	✓				✓	exp + const	without error
6	Grey linear regression	✓	✓			✓		exp + const	with error
7	GM(1,1,t^α)	✓	✓	✓	✓	✓		exp + const + vel + acc	with error

∗ Structural adaptability: AS_1 denotes $x^{(0)}(k) = a_0 p^k$; AS_2 denotes $x^{(0)}(k) = a_0 p^k + a_1$; AS_2 denotes $x^{(0)}(k) = a_0 p^k + a_2 k + a_1$; AS_2 denotes $x^{(0)}(k) = a_0 p^k + a_3 k^2 + a_2 k + a_1$; where a_0, a_1, a_2, a_3 and p are nonzero constants.

† Computing method: $Method_1$ denotes whitenization method; $Method_2$ denotes connotation method.

‡ Sequence tracking ability: '`exp`' denotes pure homogeneous exponential sequence; 'exp + const' denotes inhomogeneous exponential sequence with constant term; 'exp + const + vel + acc' denotes inhomogeneous exponential sequence with constant, velocity and acceleration terms.

It can be seen from Table 1 that no model can accurately predict the non-homogeneous exponential sequence with 3 perturbation terms of acceleration, velocity and constant, although GM(1,1,t^α) may track this pattern but there are inevitable errors in theory. Meanwhile, it can be seen from Table 1 that models using connotation method have stronger tracking ability than those using whitenization method.

Aiming at the deficiencies of models mentioned above, an extended intelligent **G**rey **M**odel with **V**ariable **S**peed and **S**tructure (VSSGM model for short) is proposed in this paper. Its structure is not steadfast but variable. It can not only forecast homogeneous exponential sequence but also forecast non-homogeneous exponential sequence with disturbances of constant, velocity and acceleration perturbation terms. What's more, the proposed VSSGM model employs connotation method to improve the accuracy of prediction as much as possible. And It is more general than those models listed in Table 1 except for GM(1,1,t^α), hence, it can change its structure dynamically to adapt to the variation trend of the data sequence to be predicted.

The rest of the paper is organized as follows. In Section 2, VSSGM model and its reducible form VCGM model are established, and then, parameter estimators and the final regressive reduction formula are deduced. In Section 3, properties of the model are provided, self-adapting structure is investigated, and then complete algorithm of the model is given to facilitate the application the model. In Section 4, detailed comparisons of simulative and forecasting precision among the proposed VSSGM model, VCGM model and other traditional models including GM(1,1), DGM(1,1), NDGM, SAIGM. In Section 5, simulation is performed on China's industrial added value data from year 2006 to 2015 based on the VSSGM model, and to test the accuracy of the model, prediction value of industrial added value in 2016 is calculated based on true statistical data, and the China's industrial added values from year 2017 to 2021 is forecast in the last. Conclusions are placed in Section 6.

2. VSSGM model

2.1. *Accumulated generating operator and inverse accumulated generating operator [8]*

Definition 1. Assume $X^{(0)}$ is nonnegative raw sequence $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n))$, and α is sequential operator, satisfying $X^{(0)}\alpha = X^{(1)} = (x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n))$, where $x^{(1)}(i) = \sum_{j=1}^{i} x^{(0)}(j), i = 1, 2, \cdots, n$. Then the sequential operator α is called first order accumulating generation operator(1-AGO). If accumulating generation sequential operator α is applied m times on raw sequence $X^{(0)}$, then $X^{(0)}\alpha^m = X^{(m)} = (x^{(m)}(1), x^{(m)}(2), \cdots, x^{(m)}(n))$, where $x^{(i)} = \sum_{j=1}^{i} x^{(m-1)}(j), i = 1, 2, \cdots, n$. The inverse operation of accumulating generation process is called inverse accumulating generation, which can be defined as $\beta^{(m)}x^{(0)}(i) = \alpha^{(m-1)}x^{(0)}(i) - \alpha^{(m-1)}x^{(0)}(i-1)$. The operator β is call inverse accumulating sequence operator.

Accumulating generation operation can attenuate the randomness of the raw sequence to make it more suitable for grey modeling, inverse accumulating generation operation is the inverse process of the accumulating generation operation, and it plays a role in regression of accumulating generation operator.

2.2. *VSSGM model and its parameter estimation*

Definition 2. Let $X^{(1)}$ be the same as that in Definition 1. A new generated data sequence $Z^{(1)}$ is obtained, namely, $Z^{(1)} = (z^{(1)}(2), z^{(1)}(3), \cdots, z^{(1)}(n))$, where $z^{(1)}(i) = 0.5 \times [x^{(1)}(i) + x^{(1)}(i-1)], i = 2, 3, \cdots, n$. $Z^{(1)}$ is the even generation sequence generated by consecutive adjacent elements in the 1-AGO sequence $X^{(1)}$.

Definition 3. Assume $X^{(0)}, X^{(1)}$ *and* $Z^{(1)}$ are defined by Definition 1 and 2. Then, the following equation

$$x^{(0)}(k) + az^{(1)}(k) = b_1k^3 + b_2k^2 + b_3k + b_4, k \in \mathbb{Z}, k \geq 2 \tag{7}$$

$$\begin{cases} x^{(0)}(k) = a_0p^k \rightarrow \text{without perturbation} \\ x^{(0)}(k) = a_0p^k + a_1 \rightarrow \text{with constant perturbation term} \\ x^{(0)}(k) = a_0p^k + a_2k + a_1 \rightarrow \text{with perturbations of constant and velocity} \\ x^{(0)}(k) = a_0p^k + a_3k^2 + a_2k + a_1 \rightarrow \text{with perturbations of constant, velocity and acceleration} \end{cases}$$

is referred to as grey model with variable speed and structure (abbreviated as VSSGM). And $-a$ is called development index, $b_i(i = 1, 2, 3, 4)$ is called grey actuating quantity. a_0, a_1, a_2, a_3 and p are constants and $a_0 \neq 0, p > 0$. It is the extended form of GM(1,1) model by adding additional terms b_1k^3, b_2k^2 and b_3k to original constant term b_4. It is characterized by alterable structure obtained by altering different combination of b_1k^3, b_2k^2, b_3k and b_4. And it can simulate not only homogeneous exponential sequence but also approximate non-homogeneous exponential sequence with acceleration, velocity and constant terms. When $b_1 = 0$, VSSGM is reducible to Grey Model with Velocity and Constant terms only, abbreviated as VCGM model.

Theorem 1. *Let $X^{(0)}, X^{(1)}$ and $Z^{(1)}$ be the same as those in Definition 3. If $\hat{a} = (a, b_1, b_2, b_3, b_4)$ is a parameter vector to be estimated and*

$$\mathbf{Y} = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \cdots \\ x^{(0)}(n) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -z^{(1)}(2) & 2^3 & 2^2 & 2 & 1 \\ -z^{(1)}(3) & 3^3 & 3^2 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -z^{(1)}(n) & n^3 & n^2 & n & 1 \end{bmatrix}, \tag{8}$$

then, the least square estimation of the VSSGM model satisfies

$$\hat{a} = (a, b_1, b_2, b_3, b_4)^{\mathrm{T}} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Y}. \tag{9}$$

Proof. Feeding all data into Eq.(7), it follows that $\mathbf{Y} = \mathbf{B}\hat{a}$, and substituting

$-az^{(1)}(k) + b_1k^3 + b_2k^2 + b_3k + b_4$

for $x^{(0)}(k)$,$k = 2, 3, \cdots, n$, gives error sequence $\varepsilon = \mathbf{Y} - \mathbf{B}\hat{a}$. Let

$$
\begin{aligned}
S &= \varepsilon^{\mathrm{T}}\varepsilon = [\mathbf{Y} - \mathbf{B}\hat{a}]^{\mathrm{T}}[\mathbf{Y} - \mathbf{B}\hat{a}] \\
&= \sum_{i=2}^{n}[x^{(0)}(k) + az^{(1)}(k) - b_1k^3 - b_2k^2 - b_3k - b_4]^2,
\end{aligned}
$$

then parameters a, b_1, b_2, b_3, b_4 which make the S minimum should satisfy

$$
\begin{cases}
\dfrac{\partial S}{\partial a} = 2\sum\limits_{k=2}^{n}[x^{(0)}(k) + az^{(1)}(k) - b_1k^3 - b_2k^2 - b_3k - b_4]z^{(1)}(k) = 0, \\
\dfrac{\partial S}{\partial b_1} = -2\sum\limits_{k=2}^{n}[x^{(0)}(k) + az^{(1)}(k) - b_1k^3 - b_2k^2 - b_3k - b_4]k^3 = 0, \\
\dfrac{\partial S}{\partial b_2} = -2\sum\limits_{k=2}^{n}[x^{(0)}(k) + az^{(1)}(k) - b_1k^3 - b_2k^2 - b_3k - b_4]k^2 = 0, \\
\dfrac{\partial S}{\partial b_3} = -2\sum\limits_{k=2}^{n}[x^{(0)}(k) + az^{(1)}(k) - b_1k^1 - b_2k^2 - b_3k - b_4]k = 0, \\
\dfrac{\partial S}{\partial b_4} = -2\sum\limits_{k=2}^{n}[x^{(0)}(k) + az^{(1)}(k) - b_1k^3 - b_2k^2 - b_3k - b_4] = 0.
\end{cases}
$$

Expand the left part of the equations system, obtain

$$
\begin{cases}
a\sum\limits_{k=2}^{n}\left(z^{(1)}(k)\right)^2 - b_1\sum\limits_{k=2}^{n}z^{(1)}(k)k^3 - b_2\sum\limits_{k=2}^{n}z^{(1)}(k)k^2 - b_3\sum\limits_{k=2}^{n}z^{(1)}(k)k - b_4\sum\limits_{k=2}^{n}z^{(1)}(k) = -\sum\limits_{k=2}^{n}x^{(0)}(k)z^{(1)}(k), \\
-a\sum\limits_{k=2}^{n}z^{(1)}(k)k^3 + b_1\sum\limits_{k=2}^{n}k^6 + b_2\sum\limits_{k=2}^{n}k^5 + b_3\sum\limits_{k=2}^{n}k^4 + b_4\sum\limits_{k=2}^{n}k^3 = \sum\limits_{k=2}^{n}x^{(0)}(k)k^3, \\
-a\sum\limits_{k=2}^{n}z^{(1)}(k)k^2 + b_1\sum\limits_{k=2}^{n}k^5 + b_2\sum\limits_{k=2}^{n}k^4 + b_3\sum\limits_{k=2}^{n}k^3 + b_4\sum\limits_{k=2}^{n}k^2 = \sum\limits_{k=2}^{n}x^{(0)}(k)k^2, \\
-a\sum\limits_{k=2}^{n}z^{(1)}(k)k + b_1\sum\limits_{k=2}^{n}k^4 + b_2\sum\limits_{k=2}^{n}k^3 + b_3\sum\limits_{k=2}^{n}k^2 + b_4\sum\limits_{k=2}^{n}k = \sum\limits_{k=2}^{n}x^{(0)}(k)k, \\
-a\sum\limits_{k=2}^{n}z^{(1)}(k) + b_1\sum\limits_{k=2}^{n}k^3 + b_2\sum\limits_{k=2}^{n}k^2 + b_3\sum\limits_{k=1}^{n}k + b_4\sum\limits_{k=2}^{n}1 = \sum\limits_{k=2}^{n}x^{(0)}(k).
\end{cases}
$$

Rearrange the equations system, obtain

$$\mathbf{P} \cdot (a, b_1, b_2, b_3, b_4)^{\mathrm{T}} = \mathbf{Q}. \tag{10}$$

Where

$$
\mathbf{P} = \begin{bmatrix}
\sum_{k=2}^{n}\left(z^{(1)}(k)\right)^2 & -\sum_{k=2}^{n}z^{(1)}(k)k^3 & -\sum_{k=2}^{n}z^{(1)}(k)k^2 & -\sum_{k=2}^{n}z^{(1)}(k)k & -\sum_{k=2}^{n}z^{(1)}(k) \\
-\sum_{k=2}^{n}z^{(1)}(k)k^3 & \sum_{k=2}^{n}k^6 & \sum_{k=2}^{n}k^5 & \sum_{k=2}^{n}k^4 & \sum_{k=2}^{n}k^3 \\
-\sum_{k=2}^{n}z^{(1)}(k)k^2 & \sum_{k=2}^{n}k^5 & \sum_{k=2}^{n}k^4 & \sum_{k=2}^{n}k^3 & \sum_{k=2}^{n}k^2 \\
-\sum_{k=2}^{n}z^{(1)}(k)k & \sum_{k=2}^{n}k^4 & \sum_{k=2}^{n}k^3 & \sum_{k=2}^{n}k^2 & \sum_{k=2}^{n}k \\
-\sum_{k=2}^{n}z^{(1)}(k) & \sum_{k=2}^{n}k^3 & \sum_{k=2}^{n}k^2 & \sum_{k=2}^{n}k & \sum_{k=2}^{n}1
\end{bmatrix},
$$

$$
\mathbf{Q} = \begin{bmatrix}
-\sum_{k=2}^{n}x^{(0)}(k)z^{(1)}(k) \\
\sum_{k=2}^{n}x^{(0)}(k)k^3 \\
\sum_{k=2}^{n}x^{(0)}(k)k^2 \\
\sum_{k=2}^{n}x^{(0)}(k)k \\
\sum_{k=2}^{n}x^{(0)}(k)
\end{bmatrix}.
$$

According to the known conditions, obtain

$$\mathbf{B}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} -z^{(1)}(2) & -z^{(1)}(3) & \dots & -z^{(1)}(n) \\ 2^3 & 3^3 & \dots & n^3 \\ 2^2 & 3^2 & \dots & n^2 \\ 2 & 3 & \dots & n \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} -z^{(1)}(2) & 2^3 & 2^2 & 2 & 1 \\ -z^{(1)}(3) & 3^3 & 3^2 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -z^{(1)}(n) & n^3 & n^2 & n & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \sum_{k=2}^n \left(z^{(1)}(k)\right)^2 & -\sum_{k=2}^n z^{(1)}(k)k^3 & -\sum_{k=2}^n z^{(1)}(k)k^2 & -\sum_{k=2}^n z^{(1)}(k)k & -\sum_{k=2}^n z^{(1)}(k) \\ -\sum_{k=2}^n z^{(1)}(k)k^3 & \sum_{k=2}^n k^6 & \sum_{k=2}^n k^5 & \sum_{k=2}^n k^4 & \sum_{k=2}^n k^3 \\ -\sum_{k=2}^n z^{(1)}(k)k^2 & \sum_{k=2}^n k^5 & \sum_{k=2}^n k^4 & \sum_{k=2}^n k^3 & \sum_{k=2}^n k^2 \\ -\sum_{k=2}^n z^{(1)}(k)k & \sum_{k=2}^n k^4 & \sum_{k=2}^n k^3 & \sum_{k=2}^n k^2 & \sum_{k=2}^n k \\ -\sum_{k=2}^n z^{(1)}(k) & \sum_{k=2}^n k^3 & \sum_{k=2}^n k^2 & \sum_{k=2}^n k & \sum_{k=2}^n 1 \end{bmatrix} = \mathbf{P}, \tag{11}$$

and

$$\mathbf{B}^{\mathrm{T}}\mathbf{Y} = \begin{bmatrix} -z^{(1)}(2) & -z^{(1)}(3) & \dots & -z^{(1)}(n) \\ 2^3 & 3^3 & \dots & n^3 \\ 2^2 & 3^2 & \dots & n^2 \\ 2 & 3 & \dots & n \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \dots \\ x^{(0)}(n) \end{bmatrix} = \begin{bmatrix} -\sum_{k=2}^n x^{(0)}(k)z^{(1)}(k) \\ \sum_{k=2}^n x^{(0)}(k)k^3 \\ \sum_{k=2}^n x^{(0)}(k)k^2 \\ \sum_{k=2}^n x^{(0)}(k)k \\ \sum_{k=2}^n x^{(0)}(k) \end{bmatrix} = \mathbf{Q}. \tag{12}$$

Substituting both $\mathbf{P} = \mathbf{B}^{\mathrm{T}}\mathbf{B}$ deduced from Eq.(11) and $\mathbf{Q} = \mathbf{B}^{\mathrm{T}}\mathbf{Y}$ derived from Eq.(12) into Eq.(10), get

$$\mathbf{B}^{\mathrm{T}}\mathbf{B} \cdot (a, b_1, b_2, b_3, b_4)^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{Y}. \tag{13}$$

Therefore,

$\hat{a} = (a, b_1, b_2, b_3, b_4)^{\mathrm{T}} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Y}.$ □

2.3. Time response sequence $\hat{x}^{(1)}(t)$ and the regressive reduction sequence $\hat{x}^{(0)}(t)$

As a matter of fact, Eq.(7) cannot be used for prediction directly. Time response sequence $\hat{X}^{(1)} = \{\hat{x}^{(1)}(k), k \in \mathbb{N}, k \geq 1\}$ and corresponding regressive reduction sequence $\hat{X}^{(0)} = \{\hat{x}^{(0)}(k), k \in \mathbb{N}, k \geq 1\}$ must be deduced.

Theorem 2. *For the VSSGM model defined in Definition 3, and parameters estimation $\hat{a} = (a, b_1, b_2, b_3, b_4)^{\mathrm{T}}$ given by Theorem 1, time response sequence $\hat{x}^{(1)}(k)$ and the regressive reduction sequence $\hat{x}^{(0)}(k)$ can be given as follows:*

$$\begin{cases} \hat{x}^{(1)}(k) = m^{k-1}\hat{x}^{(1)}(1) + n\sum_{j=1}^{4} b_j \sum_{i=2}^{k} m^{k-i} i^{4-j}, k \in \mathbb{N}, k \geq 2, \\ \hat{x}^{(1)}(1) = x^{(1)}(1); \end{cases} \tag{14}$$

and

$$\begin{cases} \hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1), k \in \mathbb{N}, k \geq 2, \\ \hat{x}^{(0)}(1) = x^{(0)}(1). \end{cases} \tag{15}$$

Where $m = \frac{1-0.5a}{1+0.5a}, n = \frac{1}{1+0.5a}$, and $\hat{a} = (a, b_1, b_2, b_3, b_4)^{\mathrm{T}}$ is estimated according to Theorem 1.

Proof. Substituting $\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)$ and $\hat{z}^{(1)}(k) = 0.5(\hat{x}^{(1)}(k) + \hat{x}^{(1)}(k-1))$ into the basic form $\hat{x}^{(0)}(k) + a\hat{z}^{(1)}(k) = b_1k^3 + b_2k^2 + b_3k + b_4$ of VSSGM model (see Definition 3), obtain

$$\hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1) + 0.5a(\hat{x}^{(1)}(k) + \hat{x}^{(1)}(k-1)) = b_1k^3 + b_2k^2 + b_3k + b_4, k \in \mathbb{Z}, k \geq 2 \tag{16}$$

Solving the equation above, obtain

$$\hat{x}^{(1)}(k) = \frac{1-0.5a}{1+0.5a}\hat{x}^{(1)}(k-1) + (b_1k^3 + b_2k^2 + b_3k + b_4)\frac{1}{1+0.5a}$$
$$= m\hat{x}^{(1)}(k-1) + n(b_1k^3 + b_2k^2 + b_3k + b_4). \tag{17}$$

Proof is given using mathematical induction method.

(1) When $k = 1$, conclusion is established obviously. If $k = 2$, there exists

$$\hat{x}^{(1)}(2) = m\hat{x}^{(1)}(1) + n(b_12^3 + b_22^2 + b_32 + b_4)$$
$$= m^{2-1}\hat{x}^{(1)}(1) + n\sum_{j=1}^{4} b_j \sum_{i=2}^{2} m^{2-i}i^{4-j}.$$

(2) Suppose the conclusion is correct when $k = p \in \mathbb{N}$ and $p > 2$, namely

$$\hat{x}^{(1)}(p) = m^{p-1}\hat{x}^{(1)}(1) + n\sum_{j=1}^{4} b_j \sum_{i=2}^{p} m^{p-i}i^{4-j}. \tag{18}$$

Then, when $k = p + 1$, there exits

$$\begin{aligned}\hat{x}^{(1)}(p+1) =& m\hat{x}^{(1)}(p) + n\Big(b_1(p+1)^3 + b_2(p+1)^2 + b_3(p+1) + b_4\Big)\\
=& m\left(m^{p-1}\hat{x}^{(1)}(1) + n\sum_{j=1}^{4} b_j \sum_{i=2}^{p} m^{p-i}i^{4-j}\right) + n\Big(b_1(p+1)^3 + b_2(p+1)^2 + b_3(p+1) + b_4\Big)\\
=& m^p\hat{x}^{(1)}(1) + n\left(b_1\sum_{i=2}^{p} m^{p+1-i}i^3 + b_2\sum_{i=2}^{p} m^{p+1-i}i^2 + b_3\sum_{i=2}^{p} m^{p+1-i}i + b_4\sum_{i=2}^{p} m^{p+1-i}\right)\\
&+ n\Big(b_1(p+1)^3 + b_2(p+1)^2 + b_3(p+1) + b_4\Big)\\
=& m^p\hat{x}^{(1)}(1) + n\left(b_1\sum_{i=2}^{p+1} m^{p+1-i}i^3 + b_2\sum_{i=2}^{p+1} m^{p+1-i}i^2 + b_3\sum_{i=2}^{p+1} m^{p+1-i}i + b_4\sum_{i=2}^{p+1} m^{p+1-i}\right)\\
=& m^p\hat{x}^{(1)}(1) + n\sum_{j=1}^{4} b_j \sum_{i=2}^{p+1} m^{p+1-i}i^{4-j}.\end{aligned}$$

So, the conclusion is true when $k = p + 1$.

Therefore, the time response sequence $\hat{X}^{(1)}$ given by Theorem 2 is true. The regressive reduction sequence $\hat{X}^{(0)}$ given by Theorem 2 is automatically established because it is deduced by $\hat{X}^{(1)}$. □

3. Properties of VSSGM model

According to deducing process of VSSGM and Theorem 2, the 3 important properties can be got.

Property 1. *The proposed VSSGM model can accurately simulate and forecast any given approximate homogeneous exponential sequence $x^{(0)}(k) = up^k + sk^2 + qk + r, k \in \mathbb{N}^+$ with 3 perturbation terms, namely, the acceleration term sk^2, the velocity term qk and constant term r without error. Where u, p, s, q and r are all constants.*

Proof. Assume approximate homogeneous exponential sequence is

$$X^{(0)} = \{up + s + q + r, up^2 + 2^2s + 2q + r, \ldots, up^n + n^2s + nq + r\}, \tag{19}$$

the following trend is

$$x^{(0)}(k) = up^k + sk^2 + qk + r, \quad k = n+1, n+2, \ldots. \tag{20}$$

Then, the 1st-order accumulating generation (1-AGO) of the $X^{(0)}$ is

$$X^{(1)} = \left\{ up + s + q + r, u(p + p^2) + s(1 + 2^2) + q(1 + 2) + 2r, \ldots, u \sum_{i=1}^{n} p^i + s \sum_{i=1}^{n} i^2 + q \sum_{i=1}^{n} + nr \right\} \tag{21}$$

Hence, the even sequence generated by consecutive neighbors of $X^{(1)}$ is

$$\begin{aligned} Z^{(1)} = \left\{ z^{(1)}(2), z^{(1)}(3), \ldots, z^{(1)}(n) \right\} = & \left\{ \frac{p^2 u}{2} + pu + 3s + 2q + \frac{3r}{2}, \ldots, \right. \\ & \left. s \sum_{i=1}^{n-1} i^2 + u \sum_{i=1}^{n-1} p^i + q \sum_{i=1}^{n-1} i + \frac{n^2 s}{2} + \frac{up^n}{2} + \frac{nq}{2} + \frac{1}{2}(2n-1)r \right\} \end{aligned} \tag{22}$$

Define

$$A = \sum_{k=2}^{n} [z^{(1)}(k)]^2, \quad B = \sum_{k=2}^{n} z^{(1)}(k)k^2, \quad C = \sum_{k=2}^{n} z^{(1)}(k)k, \quad D = \sum_{k=2}^{n} z^{(1)}(k),$$

$$E = \sum_{k=2}^{n} k^4, \quad F = \sum_{k=2}^{n} k^3, \quad G = \sum_{k=2}^{n} k^2, \quad H = \sum_{k=2}^{n} k,$$

$$I = n - 1, \quad J = \sum_{k=2}^{n} x^{(0)}(k) z^{(1)}(k), \quad K = \sum_{k=2}^{n} x^{(0)}(k)k^2, \quad L = \sum_{k=2}^{n} x^{(0)}(k)k,$$

$$O = \sum_{k=2}^{n} z^{(1)}(k)k^3, \quad P = \sum_{k=2}^{n} k^6, \quad Q = \sum_{k=2}^{n} k^5, \quad R = \sum_{k=2}^{n} x^{(0)}(k),$$

$$S = \sum_{k=2}^{n} x^{(0)}(k)k^3.$$

Substituting $A, B, C, D, E, F, G, H, I, J, K, L, M, O, P, Q, R, S$ into Eq.(11) and Eq.(12) gets that

$$\mathbf{B}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} A & -O & -B & -C & -D \\ -O & P & Q & E & F \\ -B & Q & E & F & G \\ -C & E & F & G & H \\ -D & F & G & H & I \end{bmatrix}, \quad \mathbf{B}^{\mathrm{T}}\mathbf{Y} = \begin{bmatrix} -J \\ S \\ K \\ L \\ R \end{bmatrix} \tag{23}$$

therefore,

$$\hat{a} = (a, b_1, b_2, b_3, b_4)^{\mathrm{T}} = (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Y} \tag{24}$$

$$= \left(\frac{2(1-p)}{1+p}, \frac{2s(1-p)}{3(1+p)}, \frac{q(1-p)}{1+p} + s, \frac{(6r+s)(1-p)}{3(1+p)} + q, \frac{2p(r+u)}{1+p} \right)^{\mathrm{T}} \tag{25}$$

Substituting values of a, b_1, b_2, b_3, b_4 into the $\hat{x}^{(1)}(k)$ given by Theorem 2 gets that

$$\begin{aligned} \hat{x}^{(1)}(k) &= x^{(0)}(1)\left(\frac{1-0.5a}{1+0.5a}\right)^{k-1} + \frac{1}{1+0.5a} \sum_{j=1}^{4} b_j \sum_{i=2}^{k} \left(\frac{1-0.5a}{0.5a+1}\right)^{k-i} i^{4-j} \\ &= \frac{1}{3}k^3 s + \frac{1}{2}k^2(q+s) + \left(\frac{q}{2} + r + \frac{s}{6}\right)k + \frac{p\left(1-p^k\right)u}{1-p}, \quad k \geq 1, k \in \mathbb{N} \end{aligned}$$

Therefore, it follows that

$$\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1) = up^k + sk^2 + qk + r, \quad k \geq 1, k \in \mathbb{N}$$

It can be seen that $\hat{x}^{(0)}(k) = x^{(0)}(k)$ quite precisely. Therefore, VSSGM model can simulate and forecast the exponential sequence $\{up^k + sk^2 + qk + r, k \geq 1, k \in \mathbb{N}\}$ with perturbation terms of the acceleration term sk^2, the velocity term qk and constant term r without error. □

Property 2. *As a special form of VSSGM model, the VCGM model defined in Definition 3 can accurately simulate and forecast any given approximate homogeneous exponential sequence $x^{(0)}(k) = up^k + qk + r, k \in \mathbb{N}^+$ with 2 perturbation terms of the velocity term qk and constant term r without error. Where u, p, q and r are all constants.*

Proof. The proof method is similar to Property 1. So it has been omitted here.

Theorem 3. *Assume that original sequence is*

$$X^{(0)} = \{up + r, up^2 + r, up^3 + r, \cdots, up^n + r\},$$

then VSSGM model is completely equivalent to NDGM(1,1) model.

Proof. 1-AGO sequence $X^{(1)} = \{up+r, u(p^2+p)+2r, \ldots, u\sum_{i=1}^{n} p^i + nr\}$. According to the parameter estimate method of the NDGM model[20], $\hat{\beta} = (\beta_1, \beta_2, \beta_3)^{\mathrm{T}}$,and

$$Y = \begin{bmatrix} u(p^2+p)+2r \\ u(p^3+p^2+p)+3r \\ \vdots \\ a\sum_{i=1}^{n} p^i + nr \end{bmatrix}, \quad B = \begin{bmatrix} up+r & 1 & 1 \\ u(p^2+p)+2r & 2 & 1 \\ \vdots & \vdots & \vdots \\ a\sum_{i=1}^{n-1} p^i + (n-1)r & n-1 & 1 \end{bmatrix}.$$

$$\hat{\beta} = (\beta_1, \beta_2, \beta_3)^{\mathrm{T}} = (B^T B)^{-1} B^{\mathrm{T}} Y = \begin{bmatrix} p \\ (1-p)r \\ up+r \end{bmatrix}, \beta_4 = 0.$$

Then

$$\hat{x}^{(1)}(k+1) = \beta_1^k (x^{(1)}(1) + \beta_4) + \beta_2 \sum_{j=1}^{k} j\beta_1^{k-j} + \frac{1-\beta_1^k}{1-\beta_1}\beta_3 = u\sum_{i=1}^{k+1} p^i + (k+1)r.$$

Therefore,

$$\hat{x}^{(0)}(k) = \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1) = u\sum_{i=1}^{k} p^i + kr - u\sum_{i=1}^{k-1} p^i - (k-1)r = up^k + r.$$

Therefore, $\hat{x}^{(0)}(k) = x^{(0)}(k)$. □

From the above proof process and Property 1, we find that NDGM and VSSGM model can achieve unbiased simulation of non-homogeneous exponential sequence $\{up^k + r \mid k \in \mathbb{N}\}$. And when $b_1 = b_2 = 0$, VSSGM model can be simplified as DGM(1,1) model.

Theorem 4. *Assume $b_1 = b_2 = 0$ in Equation 7, then the proposed VSSGM model is reducible to SAIGM model.*

Proof. When $b_1 = b_2 = 0$, then VSSGM model becomes $x^{(0)}(k) + az^{(1)} = b_3 k + b_4$. □

Property 3. *The proposed VSSGM model can simulate linear autoregressive sequence $\{x^{(0)}(k) = b_3 k + b_4 \mid k \in \mathbb{N}, b_3 \cdot b_4 \neq 0\}$ without error.*

Algorithm 1: VSSGM

Step 1 Generate 1-AGO from original data sequence $X^{(0)}$
Input: $X^{(0)} = [x^{(0)}(1), x^{(0)}(2), \cdots, x^{(0)}(n)]$
Data: i, j: integer
Output: 1-AGO:$X^{(1)}$
for $i \leftarrow 1$ **to** n **do**
  $x^{(1)}(i) \leftarrow \sum_{j=1}^{i} x^{(0)}(j)$;
end
Step 2 Calculate the mean sequence $Z^{(1)}$
Input: $X^{(1)} = [x^{(1)}(1), x^{(1)}(2), \cdots, x^{(1)}(n)]$
Data: i: integer
Output: $Z^{(1)}$
for $i \leftarrow 1$ **to** *n-1* **do**
  $z^{(1)}(i) \leftarrow (x^{(1)}(i) + x^{(1)}(i+1))/2$;
end
Step 3 Establish the matrix B and Y
Input: $X^{(0)}, Z^{(1)}$
Data: i: integer; tmp
Output: $\mathbf{B}, \mathbf{Y}$
for $i \leftarrow$ *1* **to** *n-1* **do** `// Constitue Y`
  $\mathrm{Y} \leftarrow [\mathrm{Y}; x^{(0)}(i+1)]$ `// append by column`
end
B$\leftarrow -$Transpose(**B**), tmp $\leftarrow$ Transpose(2:n);
for $i \leftarrow$ *1* **to** *4* **do** `// Constitue Z`
  B $\leftarrow$ [**B**,tmp.^ (4-i)]; `// MATLAB syntax`
end
Step 4 Estimate parameters a, b_1, b_2, b_3 and b_4
Input: $\mathbf{B}, \mathbf{Y}$
Data: **C**
Output: a, b_1, b_2, b_3, b_4: integer
begin
  $C \leftarrow (\mathbf{B}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Y}$;
  $a \leftarrow C(1)$, $b_1 \leftarrow C(2)$, $b_2 \leftarrow C(3)$, $b_3 \leftarrow C(2)$, $b_4 \leftarrow C(3)$
end
Step 5 Obtain $\hat{X}^{(0)} = \{\hat{x}^{(0)}(k) \mid k \in \mathbb{N}\}$
Input: a, b_1, b_2, b_3, b_4: integer
Data: i, j, k, m, n, N: integer
Output: $\hat{x}^{(1)}(i), i = 2, 3, \ldots, n$
$\hat{x}^{(1)}(1) \leftarrow x^{(1)}(1)$;
$n \leftarrow (1 - 0.5a)/(1 + 0.5a); m \leftarrow 1/(1 + 0.5a)$;
for $k \leftarrow 2$ **to** N **do**
  $\hat{x}^{(1)}(k) \leftarrow m^{k-1}\hat{x}^{(1)}(1) + n \sum_{j=1}^{4} b_j \sum_{i=2}^{k} m^{(k-i)} i^{h-j}$;
end
$\hat{x}^{(0)}(1) \leftarrow x^{(0)}(1)$;
for $k \leftarrow 2$ **to** N **do**
  $\hat{x}^{(0)}(k) \leftarrow \hat{x}^{(1)}(k) - \hat{x}^{(1)}(k-1)$
end
Step 6 Check accuracy of the VSSGM
Calculate $\varepsilon(k) = x^{(0)} - \hat{x}^{(0)}(k)$,$\Delta(k) = |\varepsilon(k)|/x^{(0)}(k)$ and $\bar{\Delta} = \frac{1}{n}\sum_{k=2}^{n} \Delta(k)$

Proof. It is known from Theorem 4 that SAIGM is a special case of VSSGM model. And we know that SAIGM can precisely simulate the linear autoregressive sequence [23]. Therefore,the proposed VSSGM model can simulate linear autoregressive sequence without error.

The 3 properties show that VSSGM model can simultaneously simulate homogeneous, non-homogeneous and linear autoregressive sequence. And we can obtain conclusions as follows:

(i) When $b_1 = b_2 = b_3 = 0$ and $a \cdot b_4 \neq 0$, the proposed VSSGM model is reducible to GM(1,1) model, but the precision of solution solved by Theorem 2 is higher than $\hat{x}^{(0)}(k+1) = (1 - \mathrm{e}^{a})\left(x^{(0)}(1) - \frac{b}{a}\right)\mathrm{e}^{-ak}, k = 1, 2, \ldots,$ which is obtained by whitenization method[9]. It is not hard to prove it can simulate the homogeneous exponential sequence without error. In this sense, the GM(1,1) model has the same accuracy with DGM(1,1) model. Accordingly, DGM(1,1) model is a special case of the proposed VSSGM model.

(ii) When $b_1 = b_2 = 0$ and $a \cdot b_3 \cdot b_4 \neq 0$, the proposed VSSGM model can be reducible to SAIGM model. Therefore, VSSGM can simulate both homogeneous exponential sequence $\{up^k \mid k \in \mathbb{N}\}$ and non-homogeneous exponential sequence $\{up^k + r \mid k \in \mathbb{N}\}$. And when $a = b_1 = b_2 = 0$, $b_3 \cdot b_4 \neq 0$, VSSGM can be simplified as single variable linear autoregressive model. And it is known from Theorem 3 that NDGM model can simulate and forecast the same sequence $\{up^k + r \mid k \in \mathbb{N}\}$ with SAIGM model. Therefore, under this circumstance, VSSGM is equivalent to NDGM model.

(iii) When $a \cdot b_3 \neq 0$, $b_1 = b_2 = b_4 = 0$, the proposed VSSGM model can be reducible to NGM(1,1,k) model[2], which can simulate approximate homogeneous exponential sequence.

It can be determined from conclusions and properties mentioned above that VSSGM model is not simply a prediction model. It unifies traditional grey prediction model with the homogeneous/non-homogenous exponential model and single variable linear autoregressive model. In reality, proposed VSSGM model includes 4 different models: DGM(1,1) model, NDGM model, SAIGM model, linear regressive model and non-homogeneous model VCGM(see Definition 3), a simplified form of VSSGM. Hence, the proposed VSSGM model has better generality and performance than above mentioned models. It has variable speed and structure which can simulate any given exponential sequence $\mathrm{e}^{c_3 k} + c_2 k^2 + c_1 k + c_0$ with disturbing acceleration term $c_2 k^2$, velocity term $c_1 k$ and constant term c_0. In the course of practical application, VSSGM model can automatically adjust parameters of a, b_1, b_2, b_3 and b_4, and select a relatively suitable structure to simulate and forecast the future developing trend for an unknown system, as shown in Fig. 2.

According to above deducing process, modeling process or algorithm of VSSGM model is summed up as five steps, and the pseudo code is illustrated in algorithm 1.

In order to simplify calculation, MATLAB and Origin can be adopted to help to solve the VSSGM model. Parameters of a, b_1, b_2, b_3 and b_4 can be figured out through running the designed program, further, simulation values $\hat{x}^{(0)}(i)$ and error values $\varepsilon(i)$ are worked out, including the future data $\hat{x}^{(0)}(n+t)$.

4. Comparisons of simulative and forecasting precision

In this section, 5 classical sequences will be adopted to test the simulative and forecasting precision of the proposed VSSGM model. Meanwhile, VCGM (see Definition 3), GM(1,1),DGM(1,1)(see [19]), NDGM(see [20]) and SAIGM(see [23]) will simulate and predict the same 5 sequences. Simulative and predictive errors will be compared among them. Five typical sequences are given as follows:

(i) Homogeneous exponential sequence

$$X1: \quad x^{(0)} = 0.9 \times 1.8^k, k = 1, 2, \ldots, 15.$$

(ii) Non-homogeneous exponential sequence with constant term

$$X2: \quad x^{(0)} = 1.2 \times 1.9^k + 1.5, k = 1, 2, \ldots, 15.$$

(iii) Non-homogeneous exponential sequence with velocity term and constant term

$$X3: \quad x^{(0)} = 1.6 \times 2.1^k + 0.5k + 1.2, k = 1, 2, \ldots, 15.$$

(iv) Non-homogeneous exponential sequence with acceleration,velocity and constant terms

$X4: \quad x^{(0)} = 1.2 \times 1.5^k + 0.3k^2 + 1.5k + 1.1, k = 1, 2, \ldots, 15.$

(v) Random number sequence including 15 data between 30 and 80.

$X5:$ 78.367, 35.079, 40.814, 48.922, 58.045, 66.443, 77.873, 68.221, 73.012, 79.344, 69.465, 77.556, 68.972, 56.917, 79.774

Firstly, models of VSSGM, VCGM, GM(1,1), DGM(1,1), NDGM, SAIGM will be constructed based on the first 10 data in each sequence ($k = 1, 2, \ldots, 10$). Then, these models will be used for forecasting the last 5 numbers of the sequence ($k = 11, 12, 13, 14, 15$). Their simulative and predictive performances will be compared. Modeling procedure is given firstly followed by precision analyses on simulation and prediction among these models. Concise results will be given and analyzed.

(i) The first 10 numbers of $X1$ are 1.62,2.916,5.249,9.448,17.006,30.611,55.1,99.180,178.523,321.342.

The 1-AGO $X1^{(1)}$ = {1.62,4.536,9.785,19.233,36.239,66.850,121.95,221.129,399.653,720.995}.

The mean sequence $Z1^{(1)}$ = {3.078,7.160,14.509,27.736,51.544,94.400,171.539,310.391,560.324}.

For different models, parameter estimators are worked out as shown in Table 2. Its simulative and predictive curves are shown in Fig. 3, and the corresponding relative error curves are shown in Fig. 4.

Table 2
Parameter estimators of six models for homogeneous exponential sequence $X1$.

Model	Estimators of parameters	Structures of models
GM(1,1)	$\hat{a}_1 = (-0.57143, 1.1571)^T$	$\hat{x}^{(0)}(k) - 0.57143z^{(1)}(k) = 1.1571$
DGM(1,1)	$\hat{\beta}_2 = (1.8, 1.62)^T$	$\hat{x}^{(1)}(k+1) = 1.8x^{(1)}(k) + 1.62$
SAIGM	$\hat{a}_3 = (-0.57143, 0.0, 1.1571)^T$	$\hat{x}^{(0)}(k) - 0.57143z^{(1)}(k) = 0.0k + 1.1571$
NDGM	$\hat{\beta}_4 = (1.8, 0.0, 1.62, 0.0)^T$	$\hat{x}^{(1)}(k+1) = 1.8x^{(1)}(k) + 0.0k + 1.62,\ \hat{x}^{(1)}(1) = x^{(1)} + 0.0$
VCGM	$\hat{a}_5 = (-0.57143, 0.0, 0.0, 1.1571)^T$	$\hat{x}^{(0)}(k) - 0.57143z^{(1)}(k) = 0.0k^2 + 0.0k + 1.1571$
VSSGM	$\hat{a}_6 = (-0.57143, 0.0, 0.0, 0.0, 1.1571)^T$	$\hat{x}^{(0)}(k) - 0.57143z^{(1)}(k) = 0.0k^3 + 0.0k^2 + 0.0k + 1.1571$

(ii) The first 10 numbers of $X2$ are 3.78,5.832,9.731,17.139,31.213,57.955,108.765,205.303,388.725,737.228.

The 1-AGO $X2^{(1)}$ = {3.78,9.612,19.343,36.481,67.695,125.650,234.414,439.717,828.442,1565.67}.

The mean sequence $Z2^{(1)}$ = {6.696,14.477,27.912,52.088,96.672,180.032,337.066,634.080,1197.056}.

For different models, parameter estimators are worked out as shown in Table 3. Its simulative and predictive curves are shown in Fig. 5, and the corresponding relative error curves are shown in Fig. 6.

Table 3
Parameter estimators of six models for non-homogeneous exponential sequence $X2$ with constant term.

Model	Estimators of parameters	Structures of models
GM(1,1)	$\hat{a}_1 = (-0.61531, -0.52513)^T$	$\hat{x}^{(0)}(k) - 0.61531z^{(1)}(k) = -0.52513$
DGM(1,1)	$\hat{\beta}_2 = (1.8887, -0.75498)^T$	$\hat{x}^{(1)}(k+1) = 1.8887x^{(1)}(k) - 0.75498$
SAIGM	$\hat{a}_3 = (-0.62069, -0.93103, 3.5379)^T$	$\hat{x}^{(0)}(k) - 0.62069z^{(1)}(k) = -0.93103k + 3.5379$
NDGM	$\hat{\beta}_4 = (1.9, -1.35, 3.78, 0.0)^T$	$\hat{x}^{(1)}(k+1) = 1.9x^{(1)}(k) - 1.35k + 3.78,\ \hat{x}^{(1)}(1) = x^{(1)}$
VCGM	$\hat{a}_5 = (-0.62069, 0.0, -0.93103, 3.5379)^T$	$\hat{x}^{(0)}(k) - 0.62069z^{(1)}(k) = 0.0k^2 - 0.93103k + 3.5379$
VSSGM	$\hat{a}_6 = (-0.62069, 0.0, 0.0, -0.93103, 3.5379)^T$	$\hat{x}^{(0)}(k) - 0.62069z^{(1)}(k) = 0.0k^3 + 0.0k^2 - 0.93103k + 3.5379$

(iii) The first 10 numbers of $X3$ are 5.060,9.256,17.518,34.317,69.046,141.426,292.874,610.366,1276.548,2674.981.

The 1-AGO $X3^{(1)}$ = {5.060,14.316,31.834,66.151,135.196,276.622,569.496,1179.862,2456.410,5131.391}.

The mean sequence $Z3^{(1)}$ = {9.688,23.075,48.992,100.673,205.909,423.059,874.679,1818.136,3793.900}.

For different models, parameter estimators are worked out as shown in Table 4. Its simulative and predictive curves are shown in Fig. 7, and the corresponding relative error curves are shown in Fig. 8.

Table 4
Parameter estimators of six models for non-homogeneous exponential sequence $X3$ with velocity and constant terms.

Model	Estimators of parameters	Structures of models
GM(1,1)	$\hat{a}_1 = (-0.70485, -1.9741)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.70485z^{(1)}(k) = -1.9741$
DGM(1,1)	$\hat{\beta}_2 = (2.0884, -3.0422)^{\mathrm{T}}$	$\hat{x}^{(1)}(k+1) = 2.0884x^{(1)}(k) - 3.0422$
SAIGM	$\hat{a}_3 = (-0.70833, -1.9757, 7.0604)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.70833z^{(1)}(k) = -1.9757k + 7.0604$
NDGM	$\hat{\beta}_4 = (2.0968, -3.059, 7.8724, -0.3848)^{\mathrm{T}}$	$\hat{x}^{(1)}(k+1) = 2.0968x^{(1)}(k) - 3.059k + 7.8724$, $\hat{x}^{(1)}(1) = x^{(1)} - 0.3848$
VCGM	$\hat{a}_5 = (-0.70968, -0.17742, -0.35161, 3.7935)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.70968z^{(1)}(k) = -0.17742k^2 - 0.35161k + 3.7935$
VSSGM	$\hat{a}_6 = (-0.70968, 0.0, -0.17742, -0.35161, 3.7935)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.17742z^{(1)}(k) = 0.0k^3 - 0.17742k^2 - 0.35161k + 3.7935$

(iv) The first 10 numbers of $X4$ are 4.7,8,12.350,17.975,25.212,34.569,46.803,63.055,85.032,115.298.

The 1-AGO $X4^{(1)}$ = {4.7,12.7,25.05,43.025,68.237,102.806,149.609,212.664,297.696,412.994}.

The mean sequence $Z4^{(1)}$ = {8.7,18.875,34.037,55.631,85.522,126.208,181.137,255.180,355.345}.

For different models, parameter estimators are worked out as shown in Table 5.Its simulative and predictive curves are shown in Fig. 9, and the corresponding relative error curves are shown in Fig. 10.

Table 5
Parameter estimators of six models for non-homogeneous exponential sequence $X4$ with acceleration,velocity and constant terms.

Model	Estimators of parameters	Structures of models
GM(1,1)	$\hat{a}_1 = (-0.30582, 7.2866)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.30582z^{(1)}(k) = 7.2866$
DGM(1,1)	$\hat{\beta}_2 = (1.361, 8.6079)^{\mathrm{T}}$	$\hat{x}^{(1)}(k+1) = 1.361x^{(1)}(k) + 8.6079$
SAIGM	$\hat{a}_3 = (-0.28276, 1.0553, 3.8262)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.28276z^{(1)}(k) = 1.0553k + 3.8262$
NDGM	$\hat{\beta}_4 = (1.3293, 1.2315, 5.6802, -0.079957)^{\mathrm{T}}$	$\hat{x}^{(1)}(k+1) = 1.3293x^{(1)}(k) + 1.2315k + 5.6802$, $\hat{x}^{(1)}(1) = x^{(1)} - 0.079957$
VCGM	$\hat{a}_5 = (-0.32897, -0.29038, 2.6452, 0.92284)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.32897z^{(1)}(k) = -0.29038k^2 + 2.6452k + 0.92284$
VSSGM	$\hat{a}_6 = (-0.4, -0.04, 0.0, 1.04, 2.76)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.4z^{(1)}(k) = -0.04k^3 + 0.0k^2 + 1.04k + 2.76$

(v) The first 10 numbers of $X5$ are 78.367,35.079,40.814,48.922,58.045,66.443,77.873,68.221,73.012,79.344.

The 1-AGO $X5^{(1)}$ = {78.367,113.446,154.260,203.182,261.227,327.670,405.543,473.764,546.776,626.120}.

The mean sequence $Z5^{(1)}$ = {95.906,133.853,178.721,232.205,294.448,366.606,439.653,510.270,586.448}.

For different models, parameter estimators are worked out as shown in Table 6.Its simulative and predictive curves are shown in Fig. 11, and the corresponding relative error curves are shown in Fig. 12.

Table 6
Parameter estimators of six models for random number sequence $X5$ including 15 data between 30 and 80.

Model	Estimators of parameters	Structures of models
GM(1,1)	$\hat{a}_1 = (-0.085393, 33.933)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.085393z^{(1)}(k) = 33.933$
DGM(1,1)	$\hat{\beta}_2 = (1.08835, 35.69)^{\mathrm{T}}$	$\hat{x}^{(1)}(k+1) = 1.08835x^{(1)}(k) + 35.69$
SAIGM	$\hat{a}_3 = (0.20474, 18.323, 15.487)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) + 0.20474z^{(1)}(k) = 18.323k + 15.487$
NDGM	$\hat{\beta}_4 = (0.79516, 17.763, 30.411, 0.15409)^{\mathrm{T}}$	$\hat{x}^{(1)}(k+1) = 0.79516x^{(1)}(k) + 17.763k + 30.411$, $\hat{x}^{(1)}(1) = x^{(1)} + 0.15409$
VCGM	$\hat{a}_5 = (-0.055184, -0.77858, 11.431, 8.0942)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) - 0.055184z^{(1)}(k) = -0.77858k^2 + 11.431k + 8.0942$
VSSGM	$\hat{a}_6 = (0.20968, -0.094617, 1.6695, 9.9324, 27.947)^{\mathrm{T}}$	$\hat{x}^{(0)}(k) + 0.20968z^{(1)}(k) = -0.094617k^3 + 1.6695k^2 + 9.9324k + 27.947$

It can be seen from Fig. 3 and Fig. 4 that the simulative and predictive values of DGM(1,1), NDGM, SAIGM, VCGM and VSSGM coincide exactly with the homogeneous exponential sequence X1. In contrast, the simulative and predictive values of GM(1,1) has gradually increasing error compared with the actual data X1. That is to say, apart from GM(1,1), models of DGM(1,1), NDGM, SAIGM, VCGM and VSSGM can simulate and predict the homogeneous exponential sequence without bias.

It can be inferred from Fig. 5 and Fig. 6 that NDGM, SAIGM, VCGM and VSSGM can simulate and forecast X2 precisely, which is one of the non-homogeneous exponential sequences with constant term, while GM(1,1) and DGM(1,1) can only simulate and predict X2 approximately. From Fig. 5 and Fig. 6, it can be seen that curves of NDGM, SAIGM, VCGM and VSSGM overlap one another exactly on the actual data sequence X2, while the curve of GM(1,1) is below those curves mentioned above, and the position of simulation and prediction curves of DGM(1,1) are just between them. This indicates that the simulative and predictive errors of DGM(1,1) is less than those of GM(1,1).

Table 7
Comparisons of simulative and predictive errors of six models for random number sequence $X5$ including 15 data between 30 and 80.

k	Actual value $X5$	GM(1,1)($a = -0.085393, b = 33.933$)		DGM(1,1)($\beta_1 = 1.0883, \beta_2 = 35.69$)		NDGM($\beta_1 = 0.79516, \beta_2 = 17.763$, $\beta_3 = 30.411, \beta_4 = 0.15409$)	
		simulative value	simulative error	simulative value	simulative error	simulative value	simulative error
		$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$
In-Sample							
$k = 1$	78.367	78.367	0	78.367	0	78.367	0
$k = 2$	35.079	42.41	20.899	42.614	21.479	32.089	8.5223
$k = 3$	40.814	46.191	13.174	46.378	13.634	43.279	6.039
$k = 4$	48.922	50.308	2.834	50.476	3.176	52.176	6.6514
$k = 5$	58.045	54.793	5.6022	54.935	5.3577	59.251	2.0772
$k = 6$	66.443	59.678	10.182	59.788	10.015	64.876	2.3581
$k = 7$	77.873	64.998	16.534	65.071	16.44	69.349	10.946
$k = 8$	68.221	70.792	3.7685	70.819	3.8087	72.906	6.8676
$k = 9$	73.012	77.103	5.6027	77.076	5.5662	75.734	3.7287
$k = 10$	79.344	83.976	5.8379	83.885	5.7237	77.983	1.7149
$\bar{\Delta}$(In)			9.3815		9.4667		5.4339
		Predictive value	Predictive error	Predictive value	Predictive error	Predictive value	Predictive error
		$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$
Out-Sample							
$k = 11$	69.465	91.462	31.666	91.296	31.428	79.772	14.837
$k = 12$	77.556	99.615	28.443	99.362	28.117	81.193	4.6901
$k = 13$	68.972	108.5	57.304	108.14	056.789	82.324	19.359
$k = 14$	56.917	118.17	107.61	117.69	106.78	83.223	46.218
$k = 15$	79.774	128.7	61.333	128.09	60.569	83.938	5.2198
$\bar{\Delta}$(Out)			57.272		56.737		18.065

k	Actual value $X5$	SAIGM ($a = 0.20474, b = 18.323$, $c = 15.487$)		VCGM ($a = -0.055184, b_1 = -0.77858$, $b_2 = 11.431, b_3 = 8.0942$)		VSSGM($a = 0.20968$, $b_1 = -0.094617, b_2 = 1.6695$, $b_3 = 9.9324, b_4 = 27.947$)	
		simulative value	simulative error	simulative value	simulative error	simulative value	simulative error
		$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$
In-Sample							
$k = 1$	78.367	78.367	0	78.367	0	78.367	0
$k = 2$	35.079	32.737	6.676	33.079	5.7021	33.762	3.7557
$k = 3$	40.814	43.278	6.0382	42.708	4.6399	42.272	3.5732
$k = 4$	48.922	51.862	6.0094	51.282	4.8236	50.649	3.5295
$k = 5$	58.045	58.851	1.3889	58.741	1.1992	58.402	0.61547
$k = 6$	66.443	64.542	2.8605	65.022	2.1381	65.137	1.9651
$k = 7$	77.873	69.177	11.167	70.059	10.035	70.533	9.425
$k = 8$	68.221	72.95	6.9321	73.78	8.1479	74.331	8.9561
$k = 9$	73.012	76.023	4.1237	76.11	4.2434	76.319	4.5297
$k = 10$	79.344	78.525	1.0325	76.972	2.9898	76.328	3.8012
$\bar{\Delta}$(In)			5.1365		4.8799		4.4612
		Predictive value	Predictive error	Predictive value	Predictive error	Predictive value	Predictive error
		$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$
Out-Sample							
$k = 11$	69.465	80.562	15.975	76.281	9.812	74.219	6.8437
$k = 12$	77.556	82.221	6.015	73.949	4.6502	69.88	9.8969
$k = 13$	68.972	83.572	21.168	69.884	1.3228	63.221	8.3376
$k = 14$	56.917	84.672	48.764	63.987	12.422	54.169	4.8287
$k = 15$	79.774	85.567	7.2623	56.154	29.609	42.663	46.521
$\bar{\Delta}$(Out)			19.837		11.563		15.286

Notes:
† $\Delta_k = \frac{|\hat{x}^{(0)}(k) - x^{(0)}(k)|}{x^{(0)}(k)} \times 100\%$
‡ $\bar{\Delta}(In) = \frac{1}{8}\sum_{k=2}^{10} \Delta_k \times 100\%$; §$\bar{\Delta}(Out) = \frac{1}{5}\sum_{k=11}^{15} \Delta_k \times 100\%$

According to Fig. 7 and Fig. 8, VCGM and VSSGM can unbiasedly simulate and forecast the non-homogeneous exponential sequence with velocity and constant terms, and NDGM and SAIGM have smaller simulative and predictive errors than GM(1,1) and DGM(1,1) models. It is can be seen from Fig. 7 and Fig. 8 that error of DGM is smaller than that of GM(1,1), but both has big simulative and predictive errors larger than 60%, and the relative percentage error of SAIGM is smaller than 10%. The relative error of NDGM has even smaller error which is less than 2%. Therefore, the proposed VSSGM and VCGM model has the highest precision than GM(1,1), DGM(1,1), NDGM and SAIGM models.

Only VSSGM model can unbiasedly simulate and predict the non-homogeneous exponential sequence with acceleration, velocity and constant terms in the general form of $\{c_1 \cdot e^k + c_2 k^2 + c_3 k + c_4\}$, as shown in Fig. 9 and Fig. 10. What is more, both GM(1,1) and DGM(1,1) have larger simulative errors than SAIGM and NDGM models, but the former two have smaller predictive errors than the latter two. Although GM(1,1) has the worst simulation performance, it has the best prediction performance among models of DGM(1,1), NDGM, SAIGM and VCGM. That is to say, good simulation performance does not mean good prediction performance, and vice versa.

For any given random sequence X5, VCGM and VSSGM have smaller simulative and predictive errors than other models mentioned above, as shown in Table 7. By comparison, it is found that the VCGM has the best prediction performance although its simulative error is not the smallest. As is shown in Fig. 11, the curves can be divided three groups by position: VCGM and VSSGM are grouped together, NDGM and SAIGM have the closest relationship, GM(1,1) and DGM(1,1) overlap almost each other. This implies that VSSGM and VCGM have the better performance than NDGM and SAIGM, and GM(1,1) and DGM(1,1) have the poorest performance in simulation and prediction. Therefore, the proposed VSSGM model shows satisfactory simulation and prediction accuracy than foregoing models, as shown in Fig. 12.

According to Figs. 3–11 and Table Table 7, the following conclusions can be drawn (see Table 8).

Table 8
Comparisons of simulation and forecasting performance among GM(1,1),DGM(1,1),NDGM,SAIGM,VCGM and VSSGM models.

Models	Sequence				
	Homogeneous exponential sequence	Non-homogeneous exponential sequence with constant term	Non-homogeneous exponential sequence with velocity and constant terms	Non-homogeneous exponential sequence with acceleration, velocity and constant terms	Random number sequence
GM(1,1)	No	No	No	No	No
DGM(1,1)	Yes	No	No	No	No
NDGM	Yes	Yes	No	No	No
SAIGM	Yes	Yes	No	No	No
VCGM	Yes	Yes	Yes	No	No
VSSGM	Yes	Yes	Yes	Yes	No

Yes: simulation/prediction without error; **No**: simulation /prediction with error.

The proposed VSSGM model in this paper can simulate and forecast any given homogeneous and non-homogeneous exponential sequence with 3 kinds of perturbation terms: acceleration, velocity and step constant. The structure of the new model can self-change to adapt to different data sequences. According to the characteristics of the original data, the proposed model can optimize its structure and parameters to minimize the simulation error. In contrast, the traditional GM models mentioned above can not simulate unbiasedly any given non-homogeneous exponential sequence with acceleration, velocity and constant terms, because their structures cannot support fitting this sequence in theory. And usually, the structures of the traditional models are steadfast although SAIGM model can adjust itself to adapt to homogeneous exponential sequence or non-homogeneous one with constant term without error. However, none of them can provide ability to fit any given sequence with perturbation of acceleration, velocity and constant terms. In contrast, the proposed VSSGM model in our work can provide more universality and adaptability to simulate any give non-homogeneous exponential sequence with 3 kinds of perturbations mentioned above simultaneously. Furthermore, unbiased simulation and prediction can be achieved in theory. In reality, the VSSGM models can forecast more kinds of sequences in real world because most of evolutionary tendencies of parameters to be predicted adhere to non-homogeneous exponential laws or approximate homogeneous exponential rules with perturbations of acceleration, velocity and constant terms. The proposed VSSGM model in our work has effectively improved the structures and prediction accuracy of the traditional grey models for it can change its structure and the corresponding model parameters dynamically according to characteristics of the modeling sequence (see Fig. 2). In general, the survey data to be simulated have non-homogeneous exponential characteristic in long development trend, mixed with three kinds of disturbances mentioned above in the developing process. By comparisons of six grey prediction models, it is concluded that the proposed VSSGM model has more accurate precision in simulation and prediction than others even for any given random sequence (see Table 7 and Fig. 12).

Table 9
China's industrial added value.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016
China's industrial added value (billion yuan)	9223.8	11169.4	13172.7	13809.5	16512.6	19514.3	20890.5	22233.7	23385.6	23650.6	23955.1

5. Prediction of China's industrial added value

With the rapid development of China's economy, China's industrial added value has increased dramatically. The volume of industrial added value in 2016 is about 3 times that in 2006, up to 23955.1 billion yuan(3774.18 billion USD). It is of great significance that a rapid developing industrial added value to increase people's living standard and promote the development of the national economy. Hence, the capacity of rational prediction on industrial added value is critical for China's government to formulate development planning. For this purpose, VSSGM, VCGM, SAIGM, NDGM, DGM(1,1) and GM(1,1) are constructed respectively in this paper to simulate industrial added values from 2006 to 2015 in China, and further to forecast the industrial added value in 2016. We will compare the VSSGM model with other five models to find the differences among them, and forecast the industrial added values from 2017 to 2021 in China. The volumes of industrial added value from 2006 to 2016 can be seen in Table 9 and Fig. 13. Data source: National Bureau of Statistics of China (http://data.stats.gov.cn/).

The research thinking and approach structure on China's industrial added value prediction in this section are shown in Fig. 14.

According to MATLAB programmed VSSGM prediction model, the values of parameters a, b_1, b_2, b_3 and b_4 are 0.17746,-44.391,869.59,-139.04,11118 respectively. And the simulated values and relative percentage errors from year 2006 to 2016 are filled in Table 10.

It can be seen from Fig. 15 and Fig. 16, that the VSSGM model has the smallest mean relative percentage error($\bar{\Delta}$). Good or bad of a simulation model cannot be judged only by $\bar{\Delta}$, for good simulation performance does not mean good prediction capacity. Therefore, it is necessary to verify the prediction accuracy of the model before applying it to predict the future data. VSSGM model, VCGM model, SAIGM model, GM(1,1) model, DGM(1,1) model and NDGM model are used to forecast the China's industrial added value in 2016, and errors are compared among them. Simulated data and errors are shown in Table 10.

From Table 10, it can be seen that the VSSGM model has minimum error among the six prediction models. Synthesizing above conclusions, we find that the VSSGM model has the smallest relative percentage error value. So, we can use the VSSGM model to forecast the volume of China's industrial added value from 2017 to 2021, see the last five lines listed in Table 10.

6. Conclusions

Traditional GM(1,1) model and its derived models with one variable first order can only forecast homogeneous or approximately homogeneous exponential sequence usually have steadfast structures, just like $\hat{x}^{(0)}(i) = ce^{-ai}$ or $\hat{x}^{(0)}(i) = ce^{-ai} + b$. Therefore, simulation accuracy based on these models may be poor when modeling data sequence do not obey homogeneous or approximately homogeneous exponential law. However, it's impossible that for a data sequence in real world follows rigorously the homogeneous exponential growing law.

In order to enhance the applicability of the prediction model, parameter adjustable VSSGM model with the alterable structure is proposed. The novel VSSGM model has the advantages of variable structure and adaptive changeable

Table 10
Comparisons of simulative and predictive errors of six models for China's industrial added value from year 2006 to 2016, and forecasting values for the next 5 years from year 2017 to 2021.

Year	Actual value of China's Industrial Added Value (billion yuan)	GM(1,1)($a = -0.089471, b = 11095$)		DGM(1,1)($\beta_1 = 1.0933, \beta_2 = 11638$)		NDGM($\beta_1 = 0.90544, \beta_2 = 3364.9,, \beta_3 = 8144.7, \beta_4 = 8.7935$)	
		simulative value	simulative error	simulative value	simulative error	simulative value	simulative error
		$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$
In-Sample							
2006	9223.8	9223.800	0	9223.800	0	9223.800	0
2007	11169.4	12469.496	11.64	12499.220	11.906	10636.562	4.7705
2008	13172.7	13636.591	3.5216	13665.659	3.7423	12995.684	1.3438
2009	13809.5	14912.923	7.9903	14940.951	8.1933	15131.724	9.5747
2010	16512.6	16308.713	1.2347	16335.255	1.074	17065.776	3.35
2011	19514.3	17835.145	8.6047	17859.676	8.479	18816.940	3.5736
2012	20890.5	19504.444	6.6349	19526.357	6.53	20402.512	2.3359
2013	22233.7	21329.983	4.0646	21348.575	3.981	21838.149	1.7791
2014	23385.6	23326.386	0.25321	23340.844	0.19138	23138.030	1.0586
2015	23650.6	25509.643	7.8604	25519.034	7.9002	24314.992	2.8092
$\bar{\Delta}$(In)			5.756		5.7774		3.3995
		Predictive value	Predictive error	Predictive value	Predictive error	Predictive value	Predictive error
		$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$
Out-Sample							
2016	23955.1	27897.245	16.456	27900.494	16.47	25380.659	5.951
$\bar{\Delta}$(Out)			16.456		16.47		5.951

Year	Actual value of China's Industrial Added Value (billion yuan)	SAIGM ($a = 0.092235, b = 3402.1, c = 5233.6$)		VCGM ($a = -0.62652, b_1 = -659.1, b_2 = -1980, b_3 = 8087.8$)		VSSGM($a = 0.17746, b_1 = -44.391, b_2 = 869.59, b_3 = -139.04, b_4 = 11118$)	
		simulative value	simulative error	simulative value	simulative error	simulative value	simulative error
		$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$
In-Sample							
2006	9223.8	9223.800	0	9223.800	0	9223.800	0
2007	11169.4	10693.941	4.2568	10586.586	5.218	11321.897	1.3653
2008	13172.7	13003.210	1.2867	12562.864	4.6295	12567.708	4.5928
2009	13809.5	15108.873	9.4093	14422.624	4.4399	14473.988	4.8118
2010	16512.6	17028.881	3.1266	16059.565	2.7436	16688.447	1.0649
2011	19514.3	18779.603	3.7649	17270.408	11.499	18916.216	3.0648
2012	20890.5	20375.965	2.463	17666.420	15.433	20910.488	0.09568
2013	22233.7	21831.577	1.8086	16504.224	25.769	22464.684	1.0389
2014	23385.6	23158.849	0.96962	12362.249	47.137	23405.896	0.086787
2015	23650.6	24369.096	3.038	2522.018	89.336	23589.398	0.25878
$\bar{\Delta}$(In)			3.3471		22.912		1.82
		Predictive value	Predictive error	Predictive value	Predictive error	Predictive value	Predictive error
		$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$	$\hat{x}^{(0)}(k)$	$\Delta_k(\%)$
Out-Sample							
2016	23955.1	25472.637	6.3349	-18215.042	176.04	23594.055	1.5072
$\bar{\Delta}$(Out)			6.3349		176.04		1.5072
Forecast							
2017						24218.474	
2018						24577.791	
2019						24600.972	
2020						24928.563	
2021						25110.798	

Notes:
† $\Delta_k = \frac{|\hat{x}^{(0)}(k) - x^{(0)}(k)|}{x^{(0)}(k)} \times 100\%$
‡ $\bar{\Delta}(In) = \frac{1}{8}\sum_{k=2}^{10} \Delta_k \times 100\%$; §$\bar{\Delta}(Out) = \frac{1}{1}\sum_{k=11}^{11} \Delta_k \times 100\%$

parameters. For any given homogeneous or non-homogeneous exponential sequence with perturbation terms of acceleration, velocity and constant noises, it can achieve high accurate simulative and predictive precision. It can serve as homogeneous exponential model, non-homogeneous model, and perhaps as single variable linear autoregressive model. This novel VSSGM model can calculate automatically to obtain the optimal model parameters and choose the most suitable and reasonable model structure to adapt the true behavior of the modeling sequence. Hence, the proposed model has better simulation performance and prediction capability than traditional models with steadfast structure.

Finally, the VSSGM model was used to simulate the China's industrial added value from 2006 to 2015, and forecast that in 2016 with some real statistic data. Results show that this novel VSSGM model has better performance in simulation and prediction than traditional models such as GM(1,1), DGM(1,1), NDGM and SAIGM models. Feasibility and applicability of the VSSGM model is verified with this example.

The proposed VSSGM model is mainly suitable for predicting the developing trend of a time dependent sequence with small quantitiy of data sample. In addition to predicting the industrial added value of China, the VSSGM model can also be used for forecasting such as the surface subsidence and deformation, resource reserves, failure and life of large scale equipments, and the demand of emergency supplies for natural disaster, etc.

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List of Figures

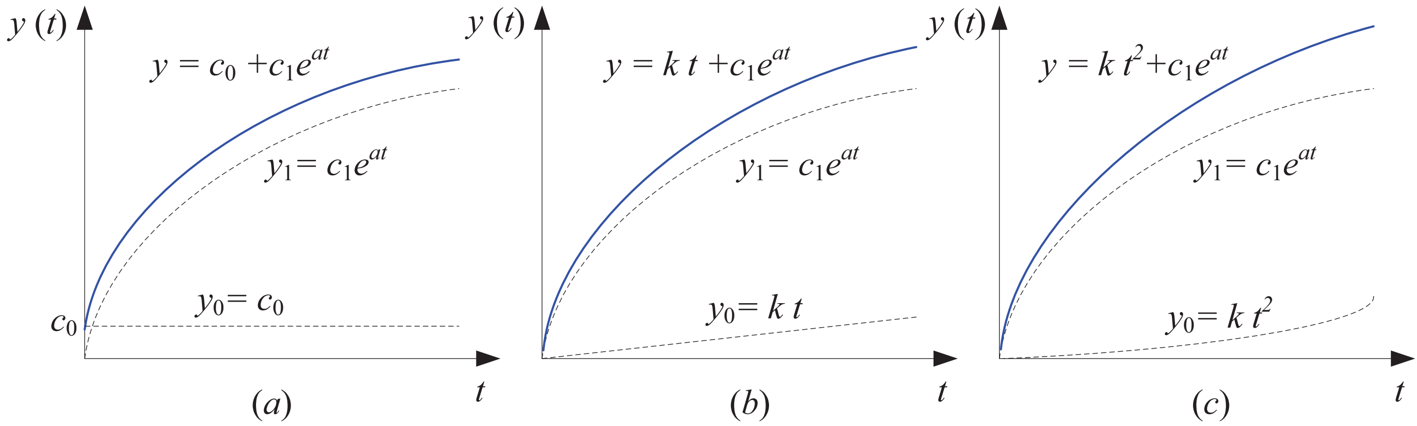


Fig. 1. Three kinds approximate exponential sequence with perturbation terms:(*a*) exponential sequence mixed with constant term; (*b*) exponential sequence mixed with velocity term; (*c*) exponential sequence with acceleration term.

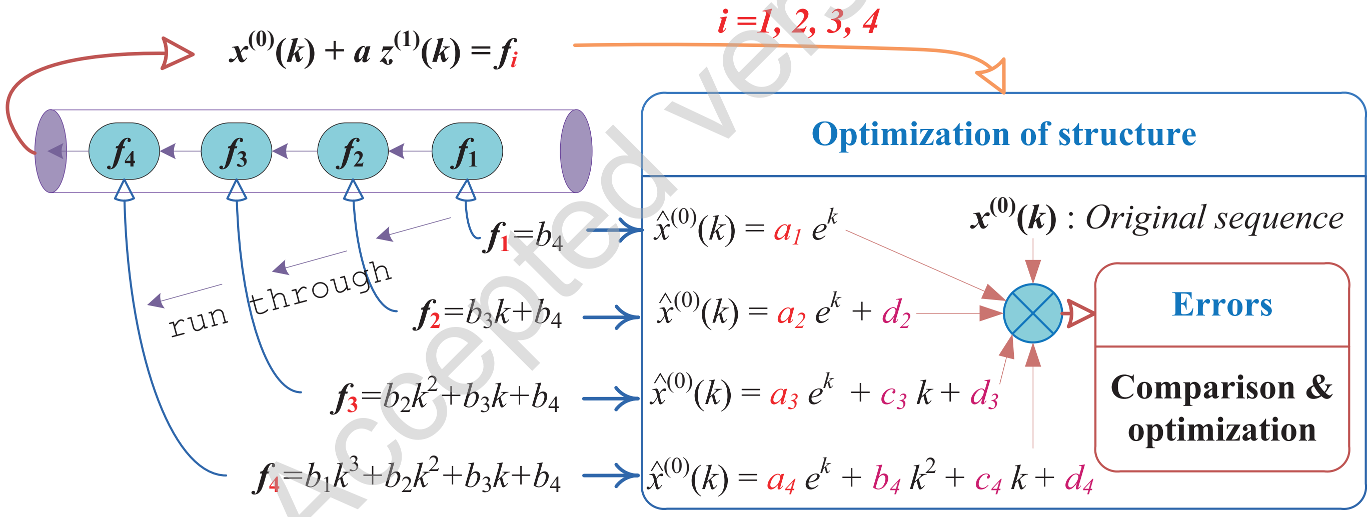


Fig. 2. Self-adpative structure of the VSSGM model and its optimization method.

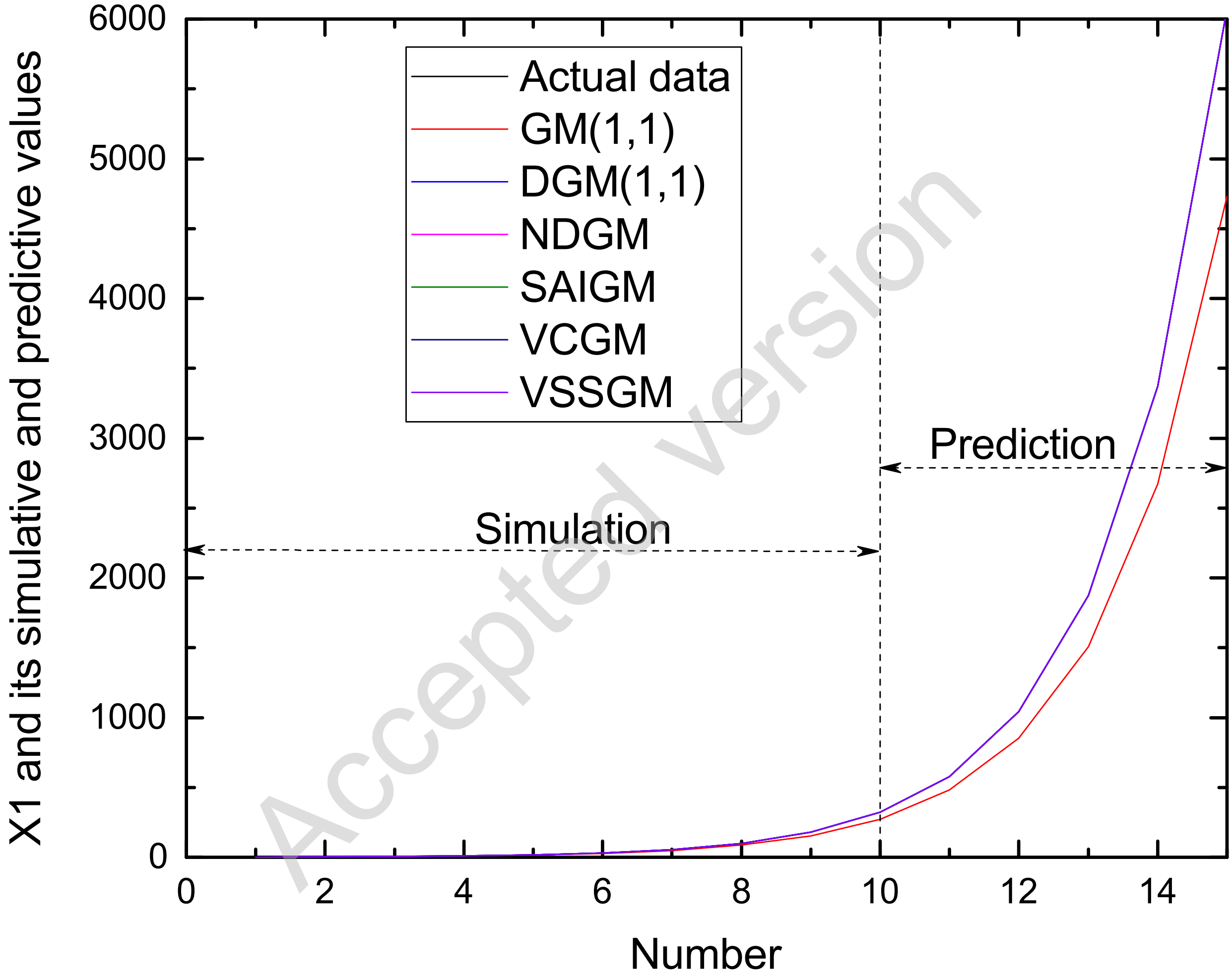


Fig. 3. X1 and its simulative and predictive curves based on 6 models.

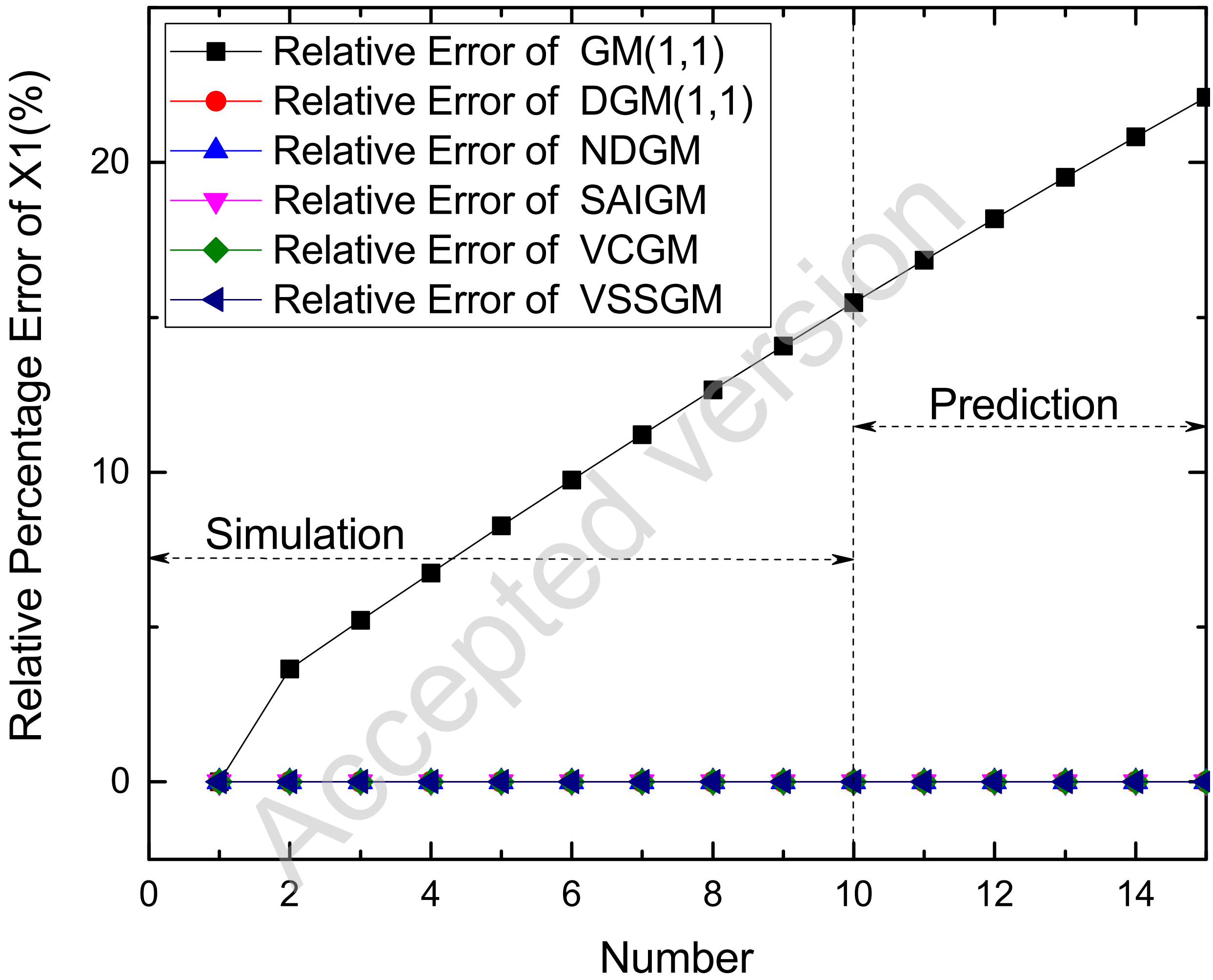


Fig. 4. Relative percentage error of X1 based on 6 models.

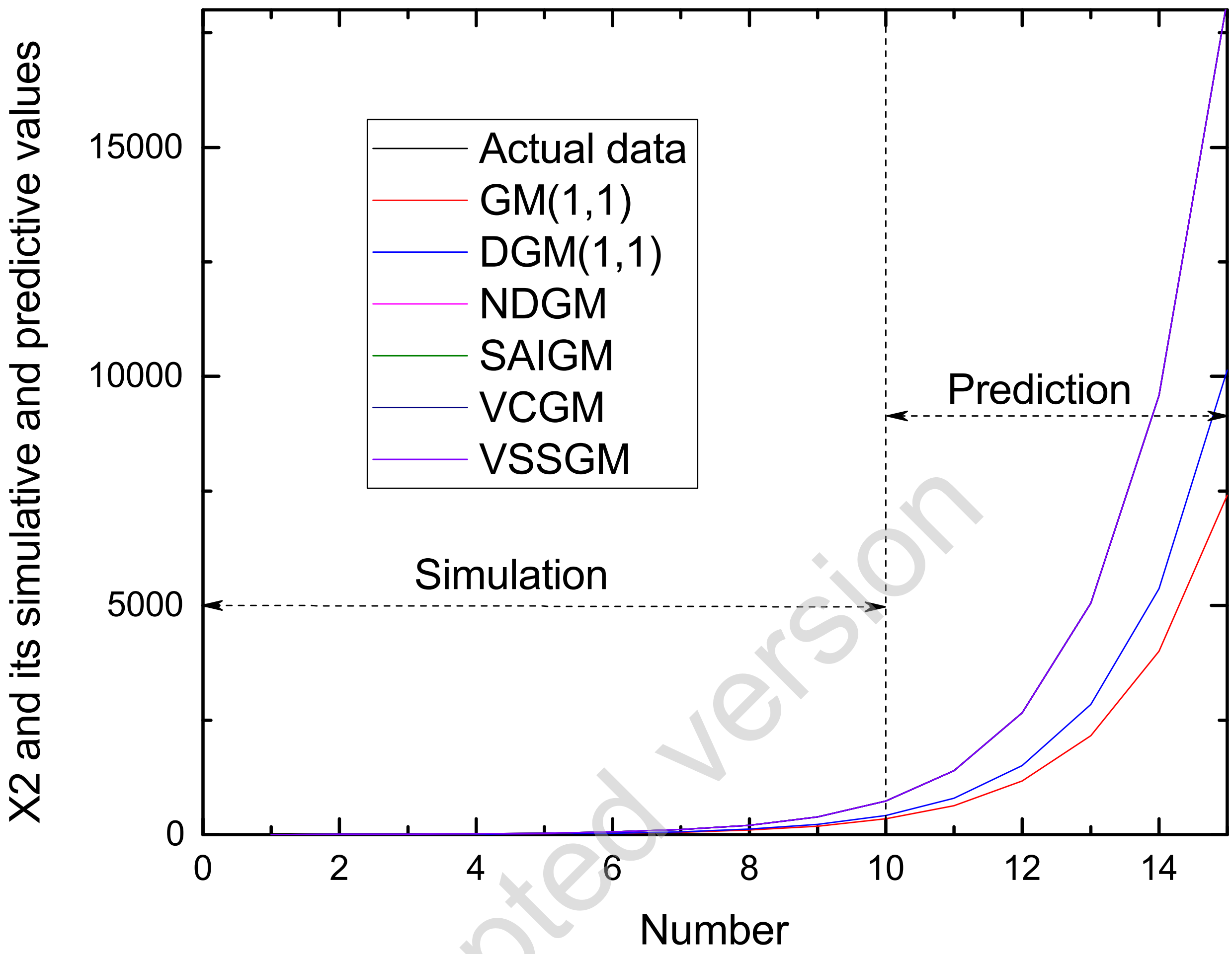


Fig. 5. X2 and its simulative and predictive curves based on 6 models.

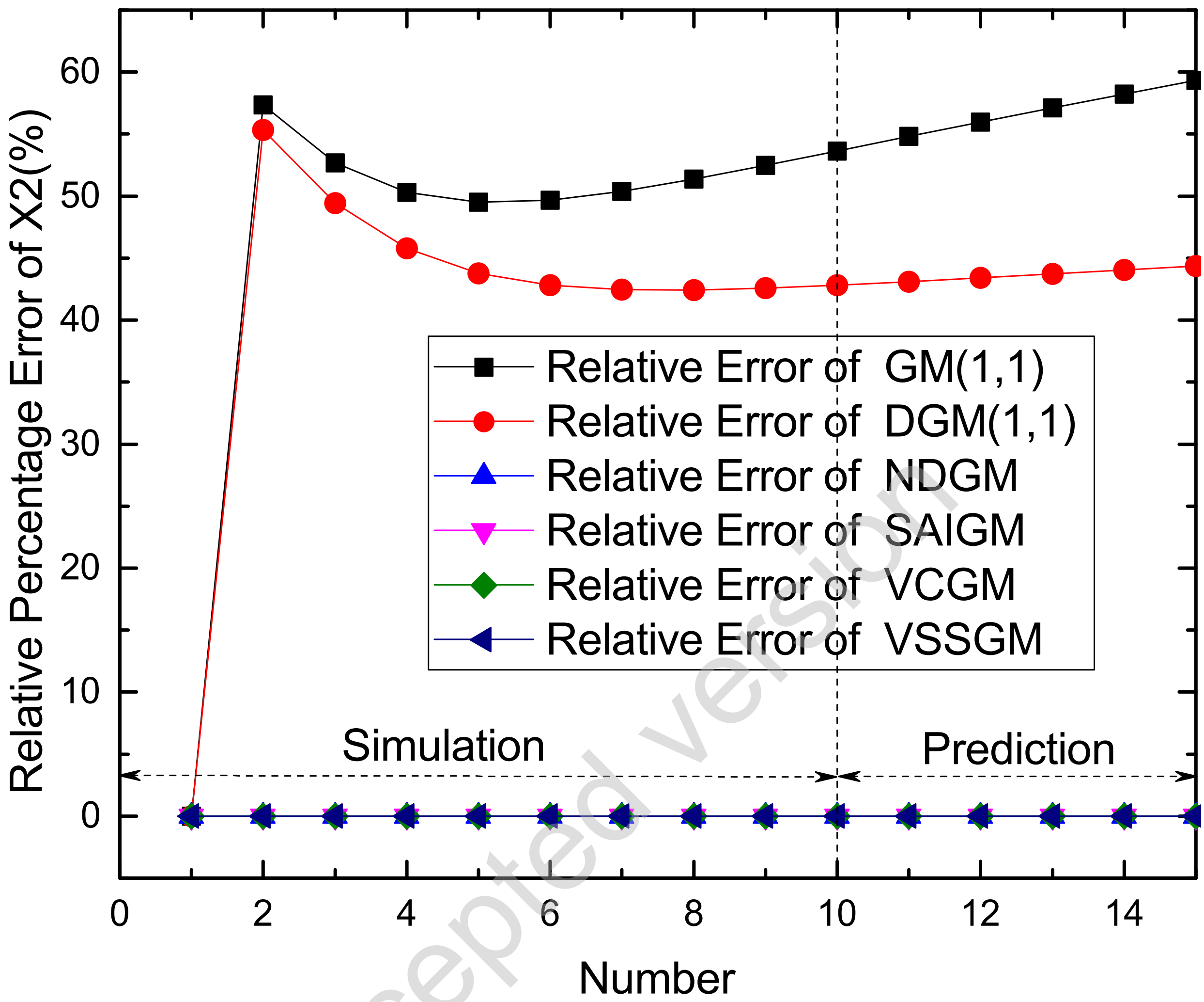


Fig. 6. Relative percentage error of X2 based on 6 models.

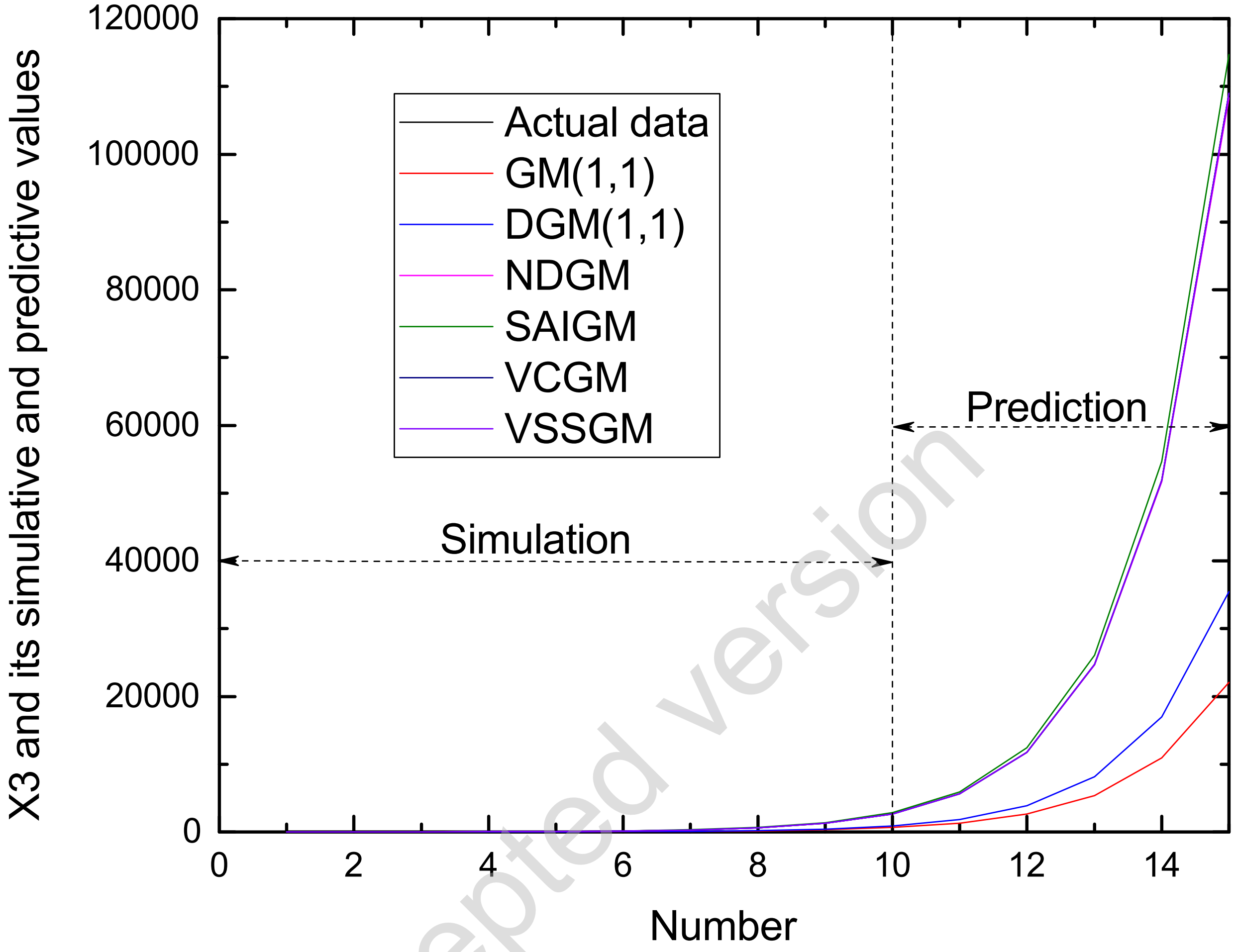


Fig. 7. X3 and its simulative and predictive curves based on 6 models.

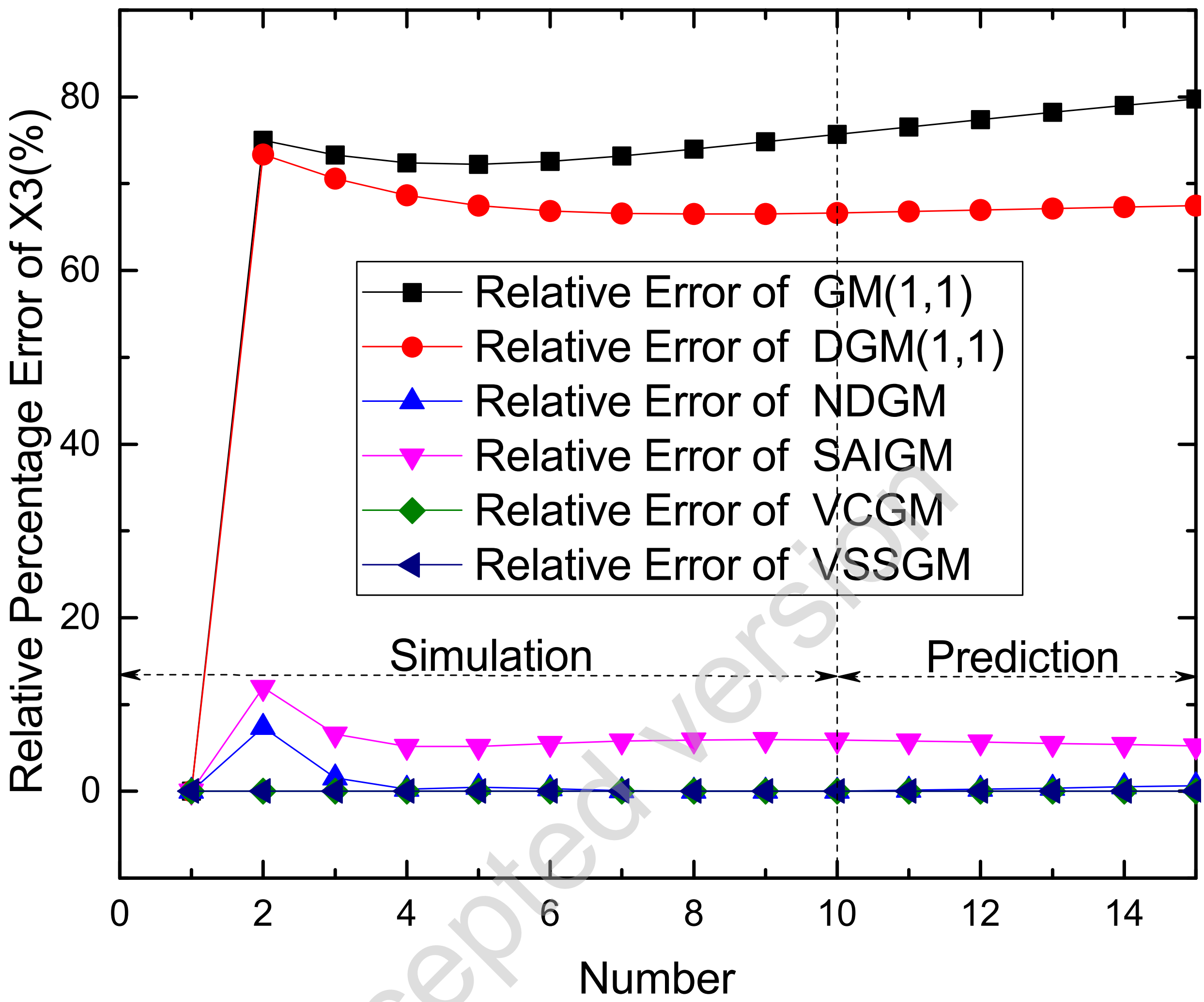


Fig. 8. Relative percentage error of X3 based on 6 models.

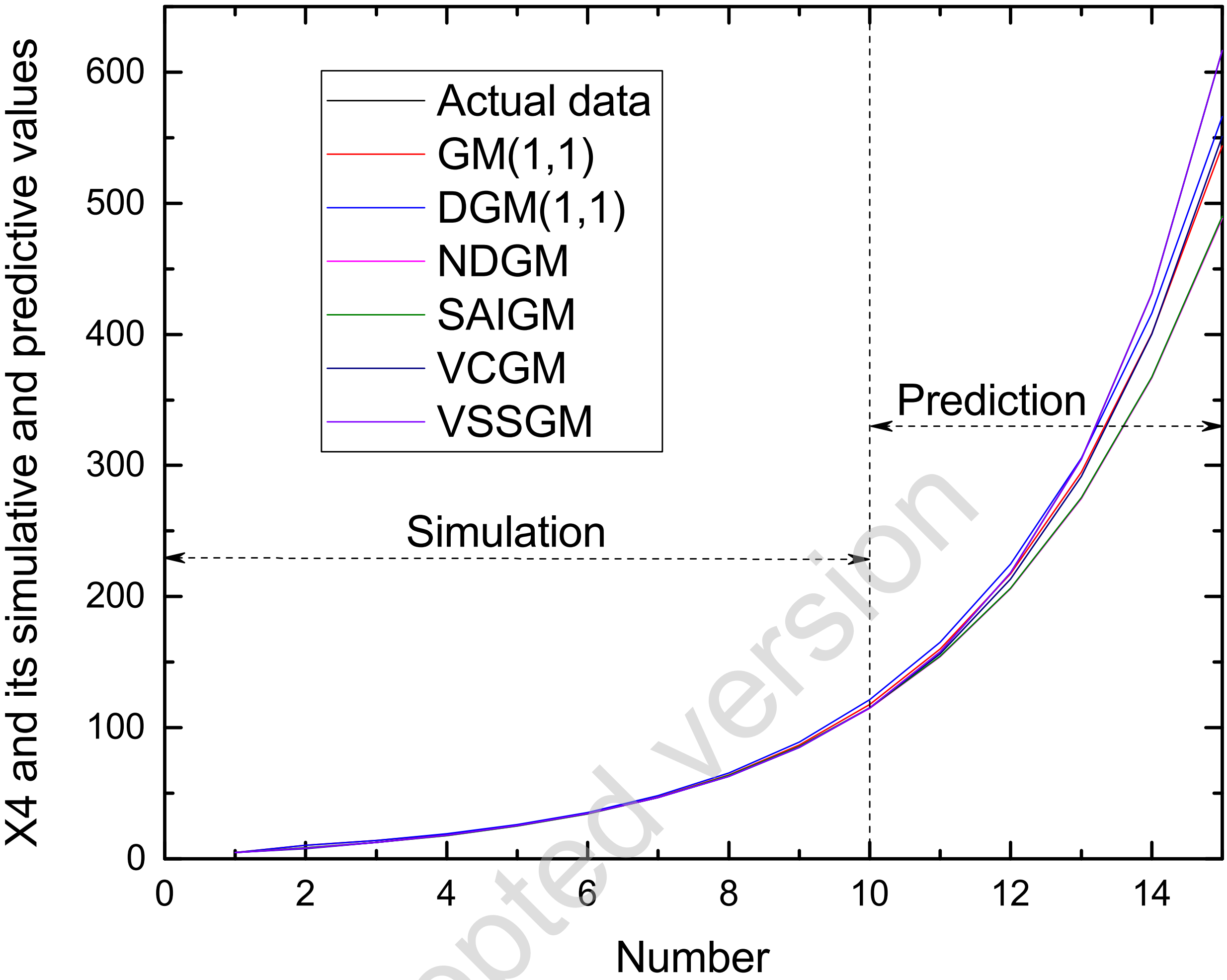


Fig. 9. X4 and its simulative and predictive curves based on 6 models.

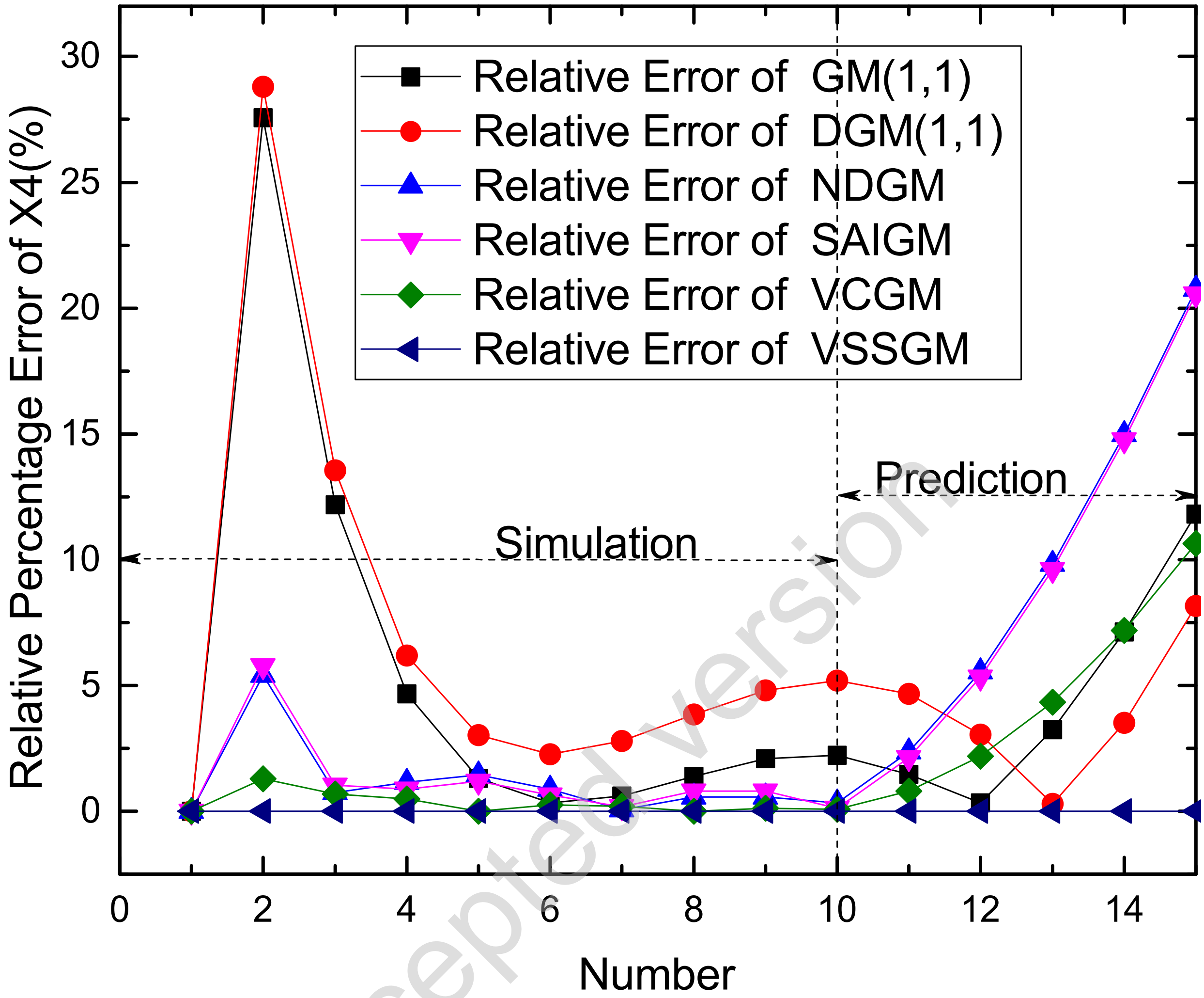


Fig. 10. Relative percentage error of X4 based on 6 models.

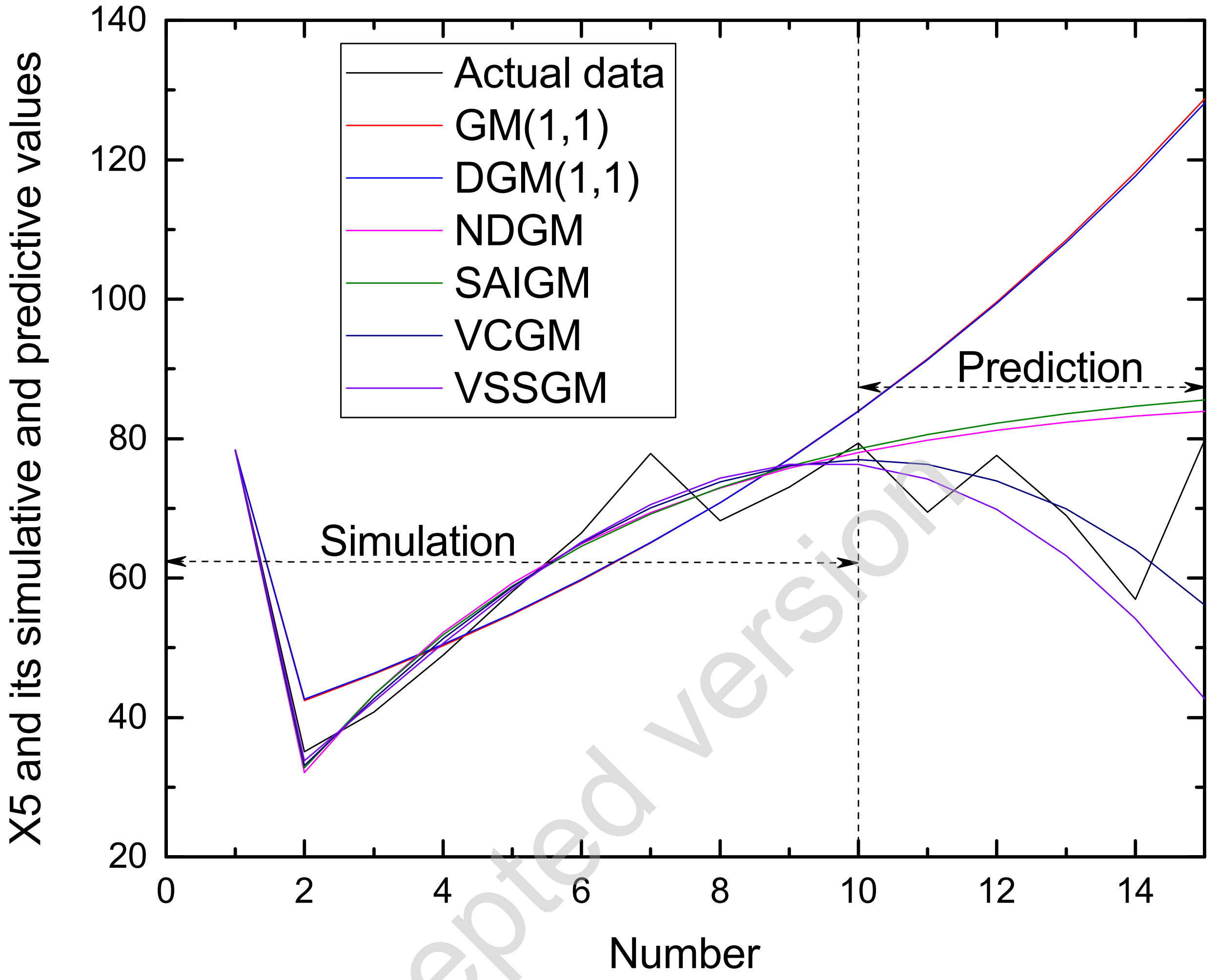


Fig. 11. X5 and its simulative and predictive curves based on 6 models.

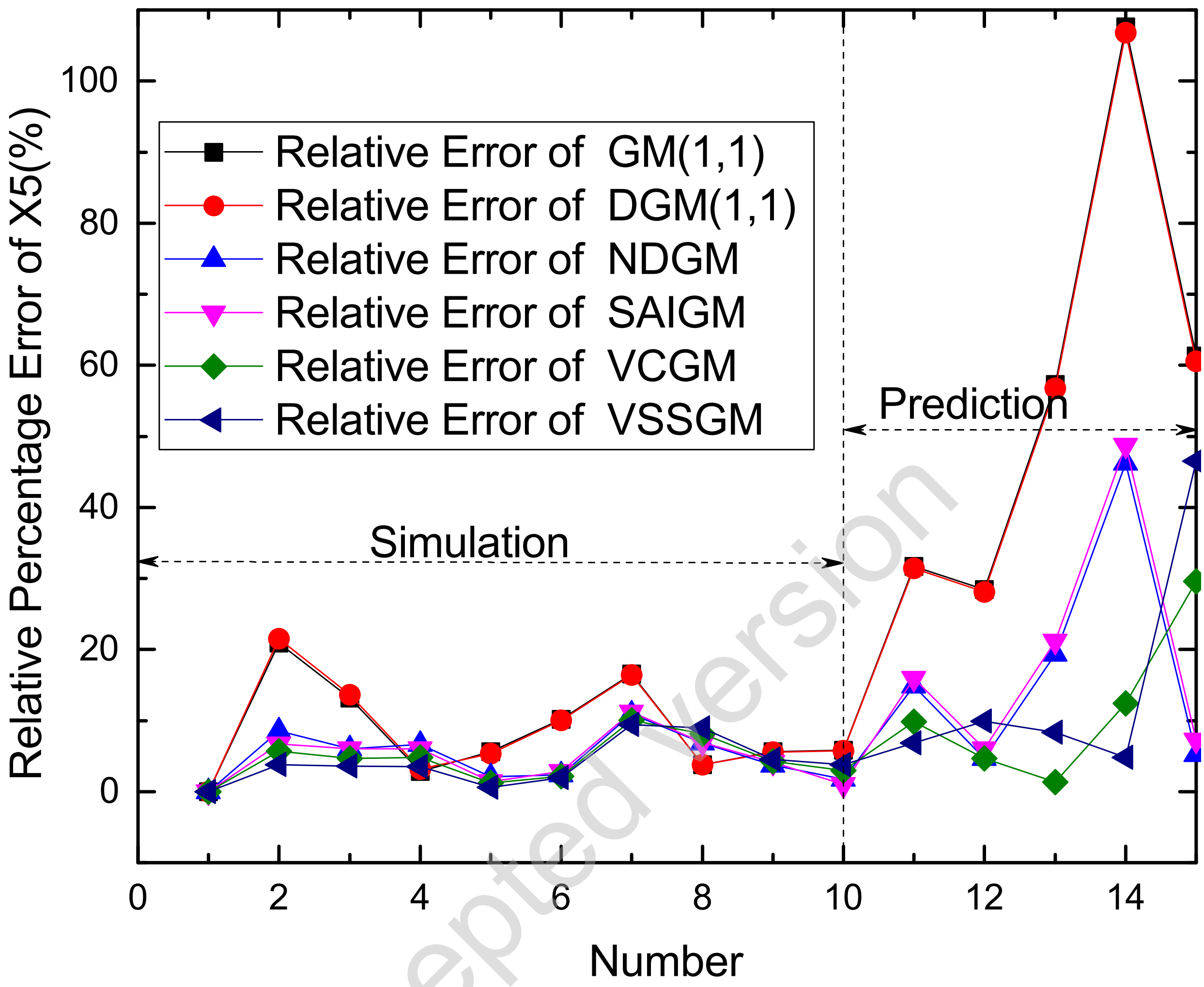


Fig. 12. Relative percentage error of X5 based on 6 models.

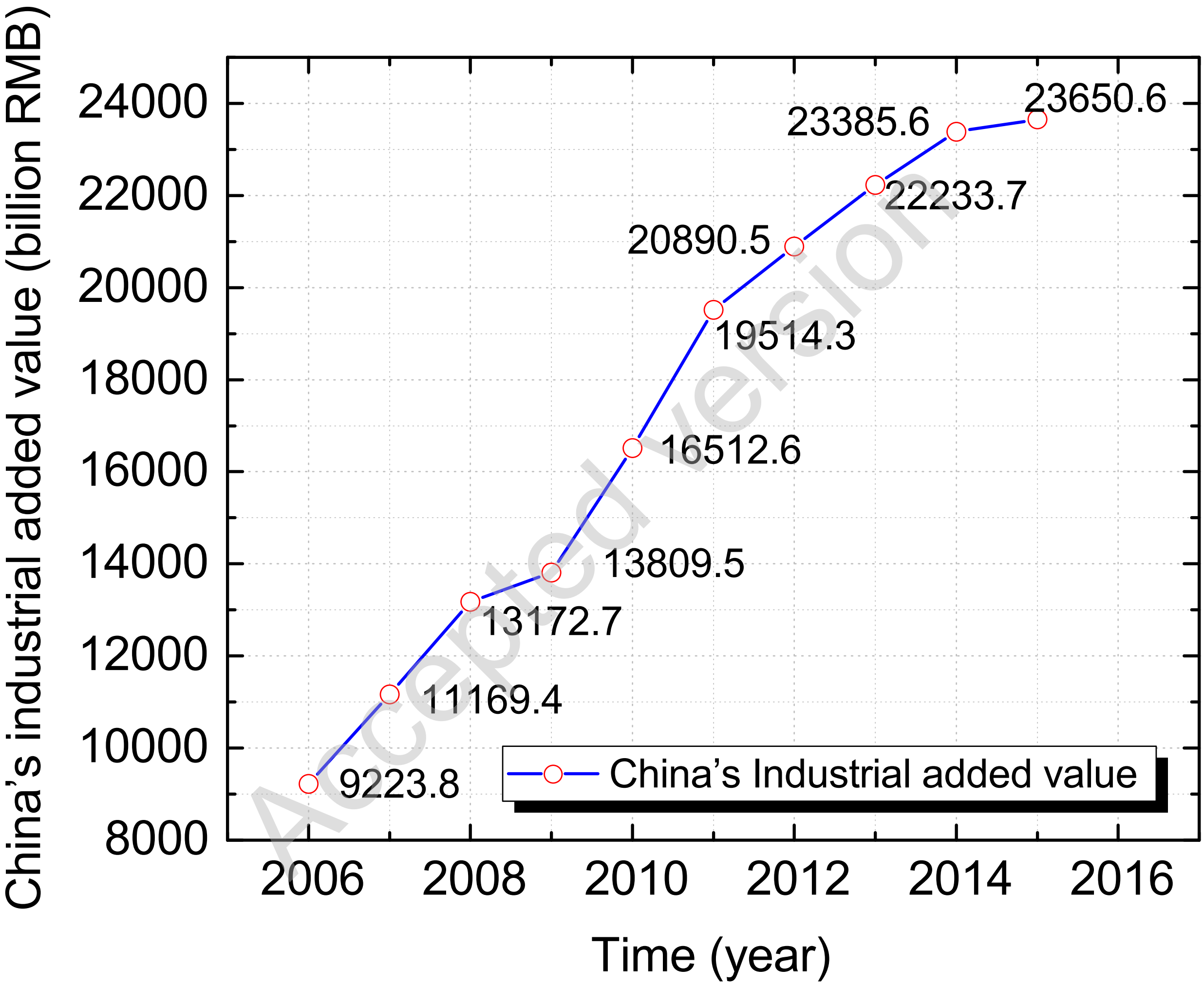


Fig. 13. Scattered broken line of China's industrial added value from 2006 to 2016.

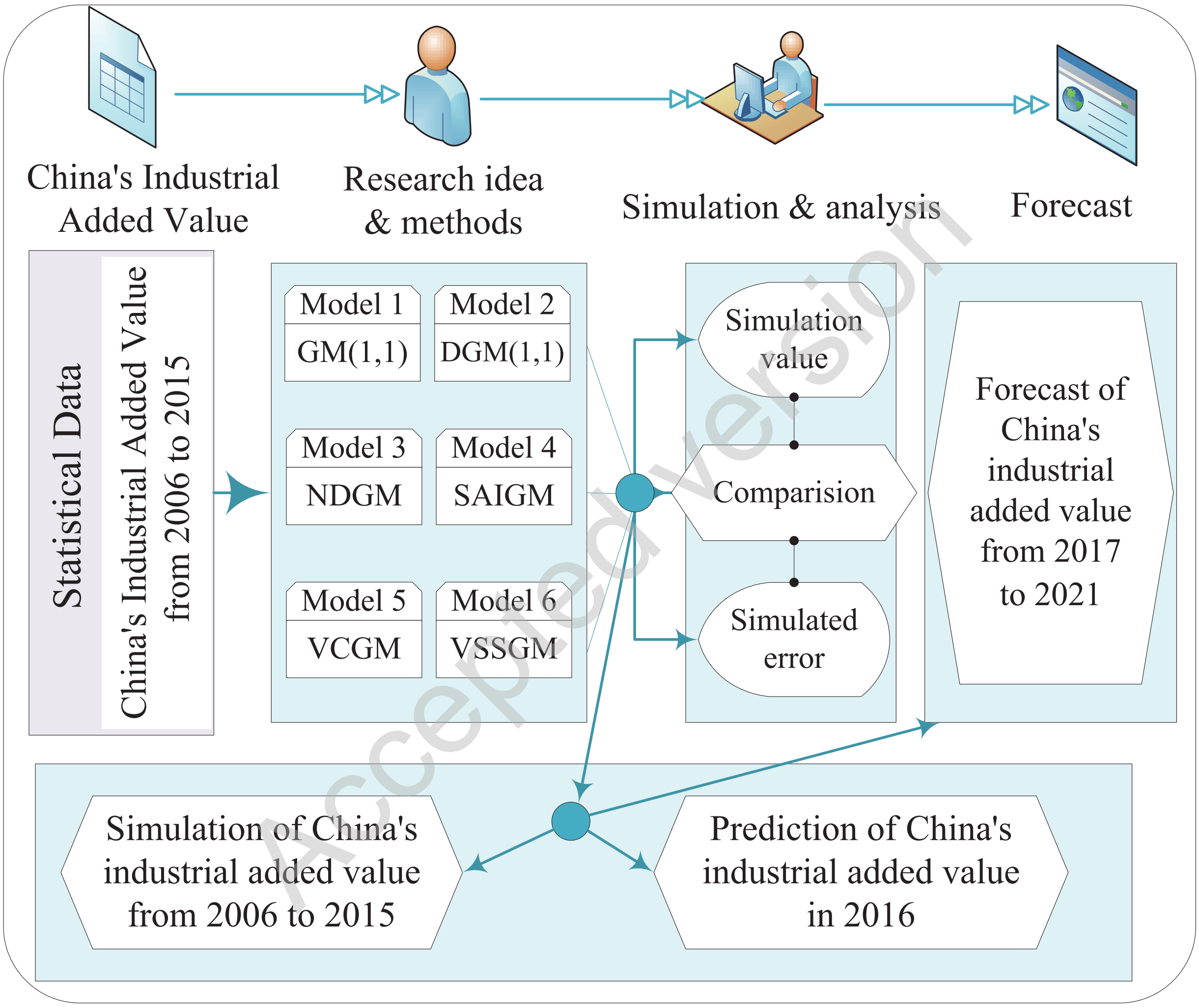


Fig. 14. Scheme of research idea and approach structure.

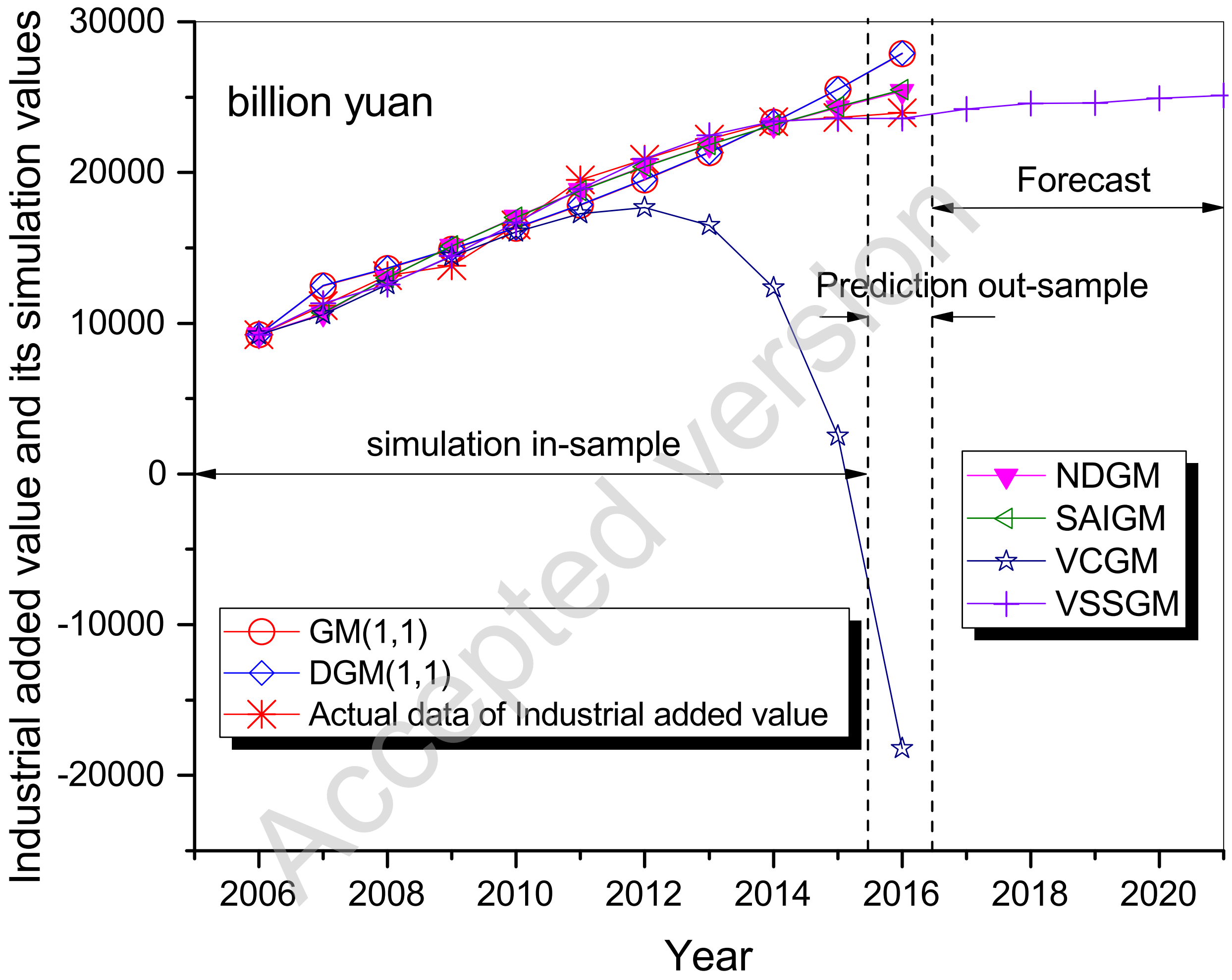


Fig. 15. Scattered broken-line of China's industrial added value and its simulative and predictive values based on 6 models.

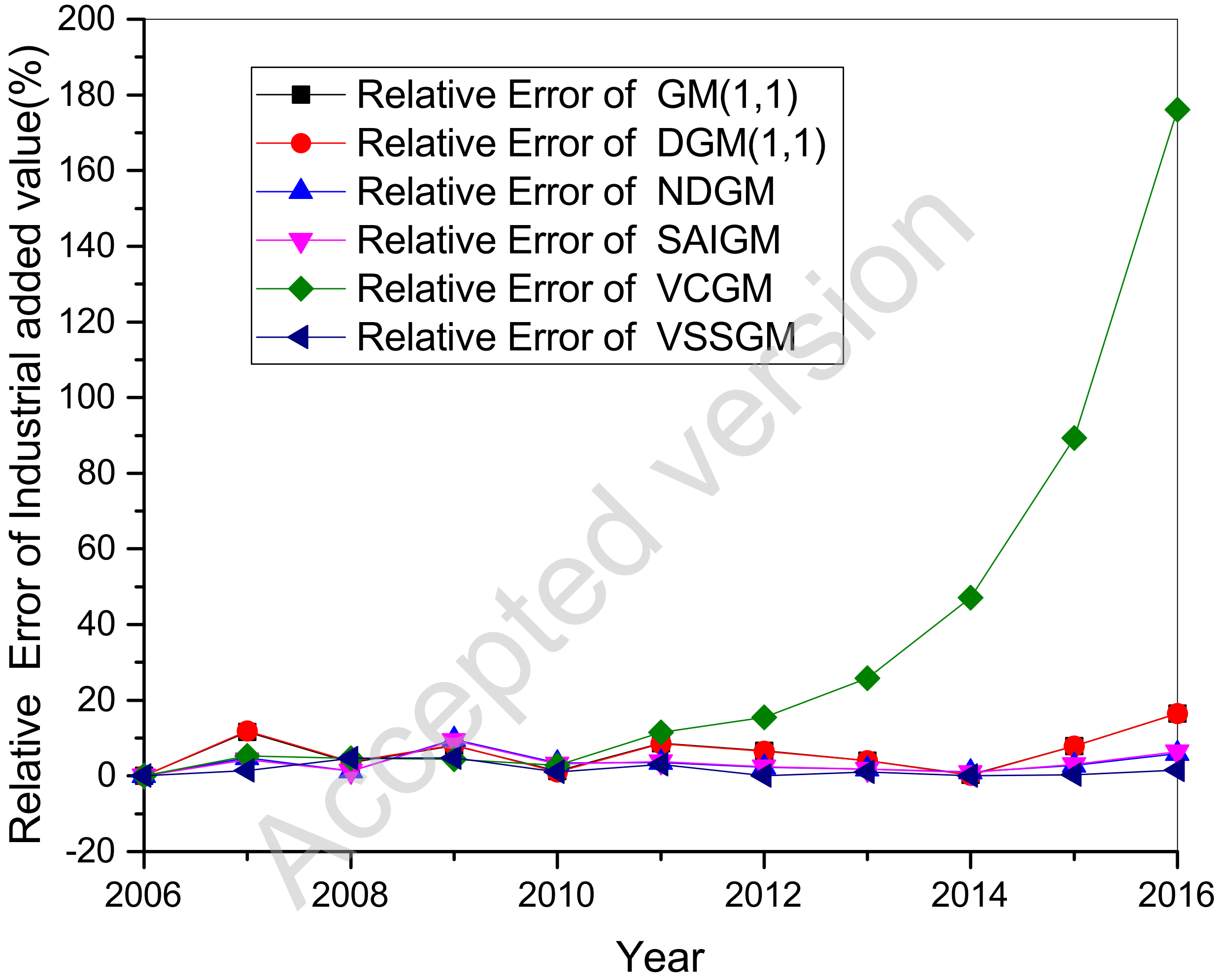
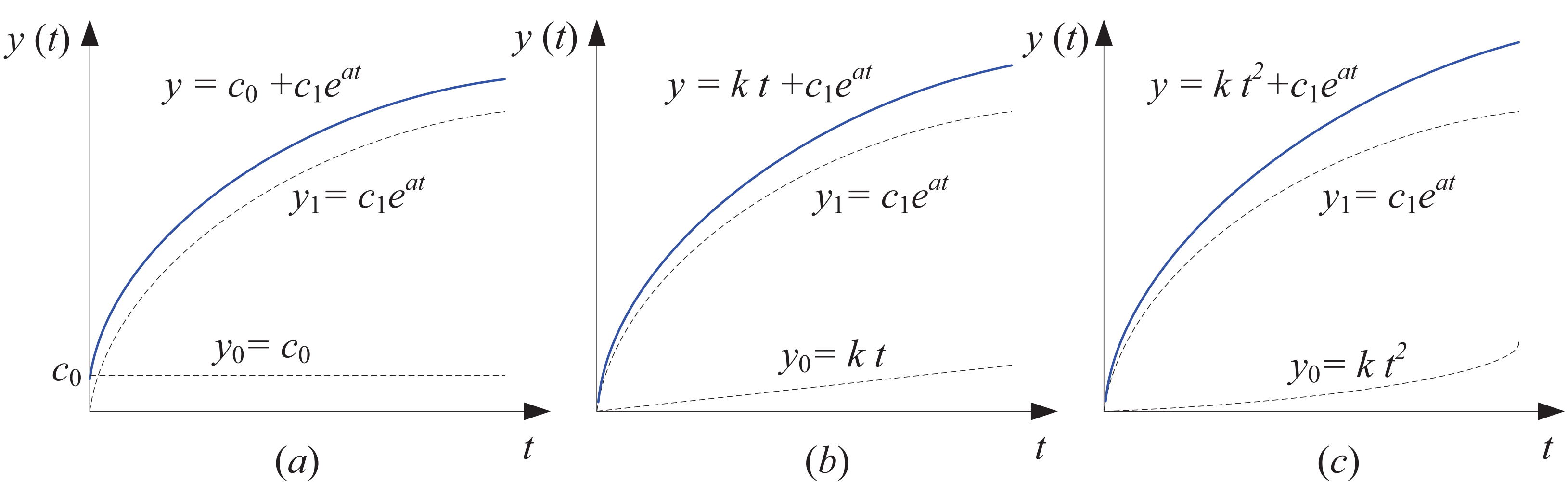
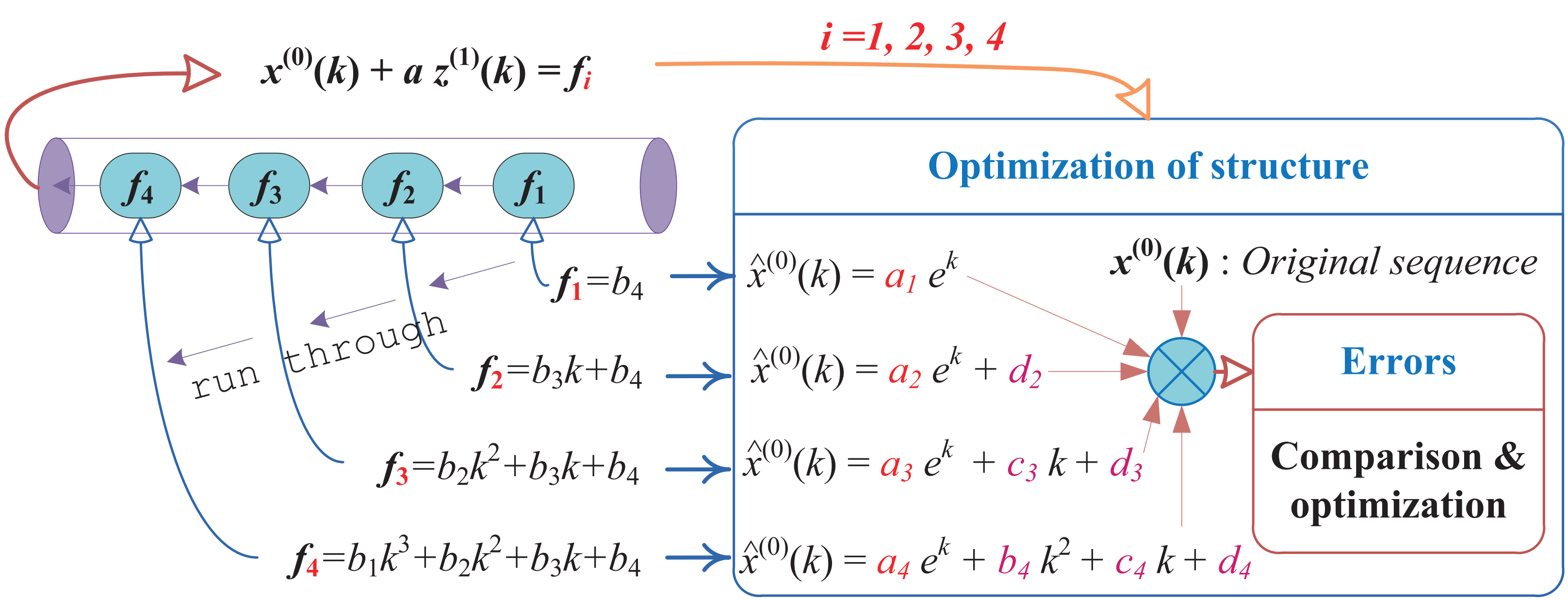


Fig. 16. Scattered broken-line of relative percentage error of China's industrial added value based on six models.

Figure

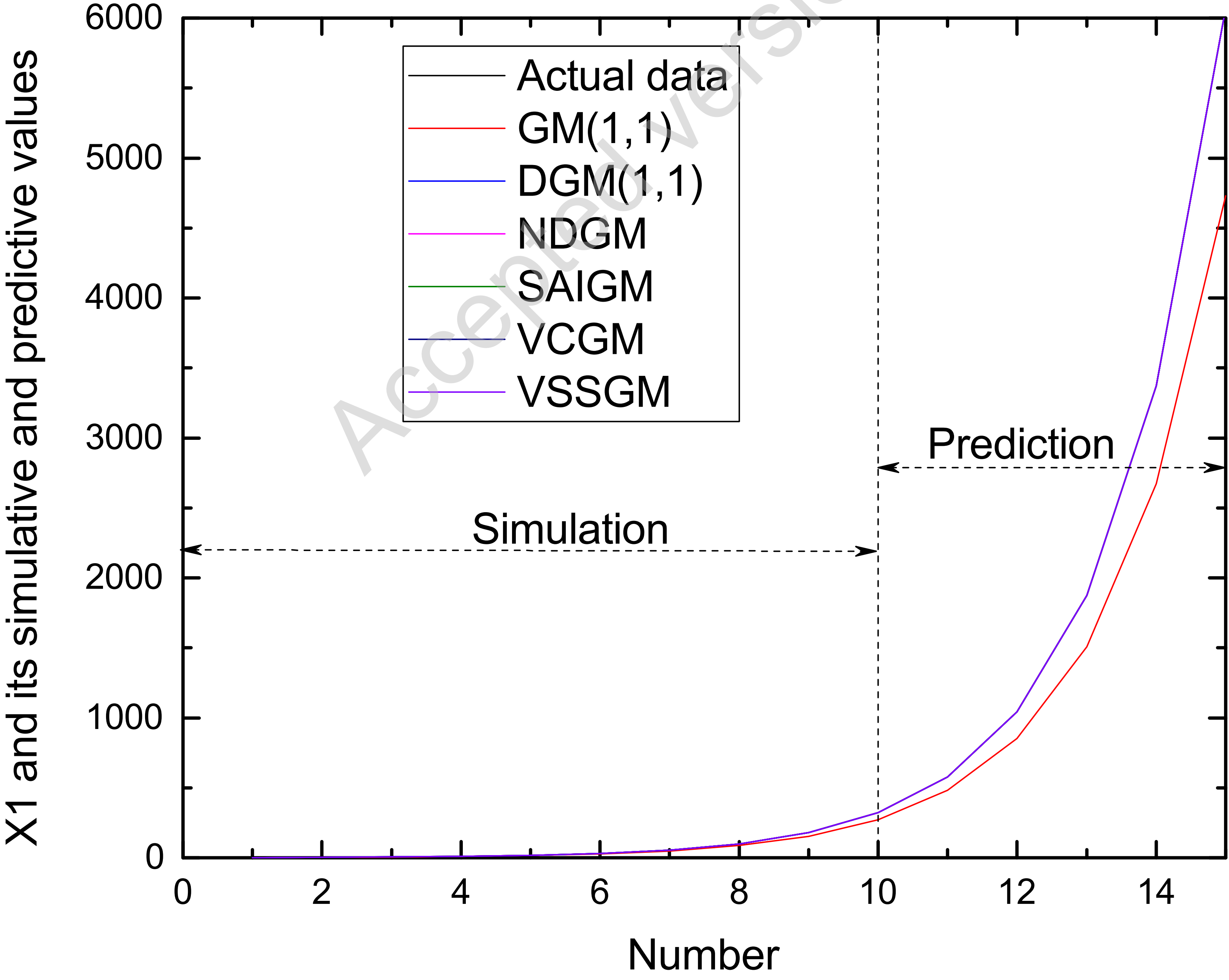

$y\,(t)$
$y = c_0 + c_1e^{at}$
$y_1 = c_1e^{at}$
$y_0 = c_0$
c_0
t
(a)
$y\,(t)$
$y = k\,t + c_1e^{at}$
$y_1 = c_1e^{at}$
$y_0 = k\,t$
t
(b)
$y\,(t)$
$y = k\,t^2 + c_1e^{at}$
$y_1 = c_1e^{at}$
$y_0 = k\,t^2$
t
(c)

Figure



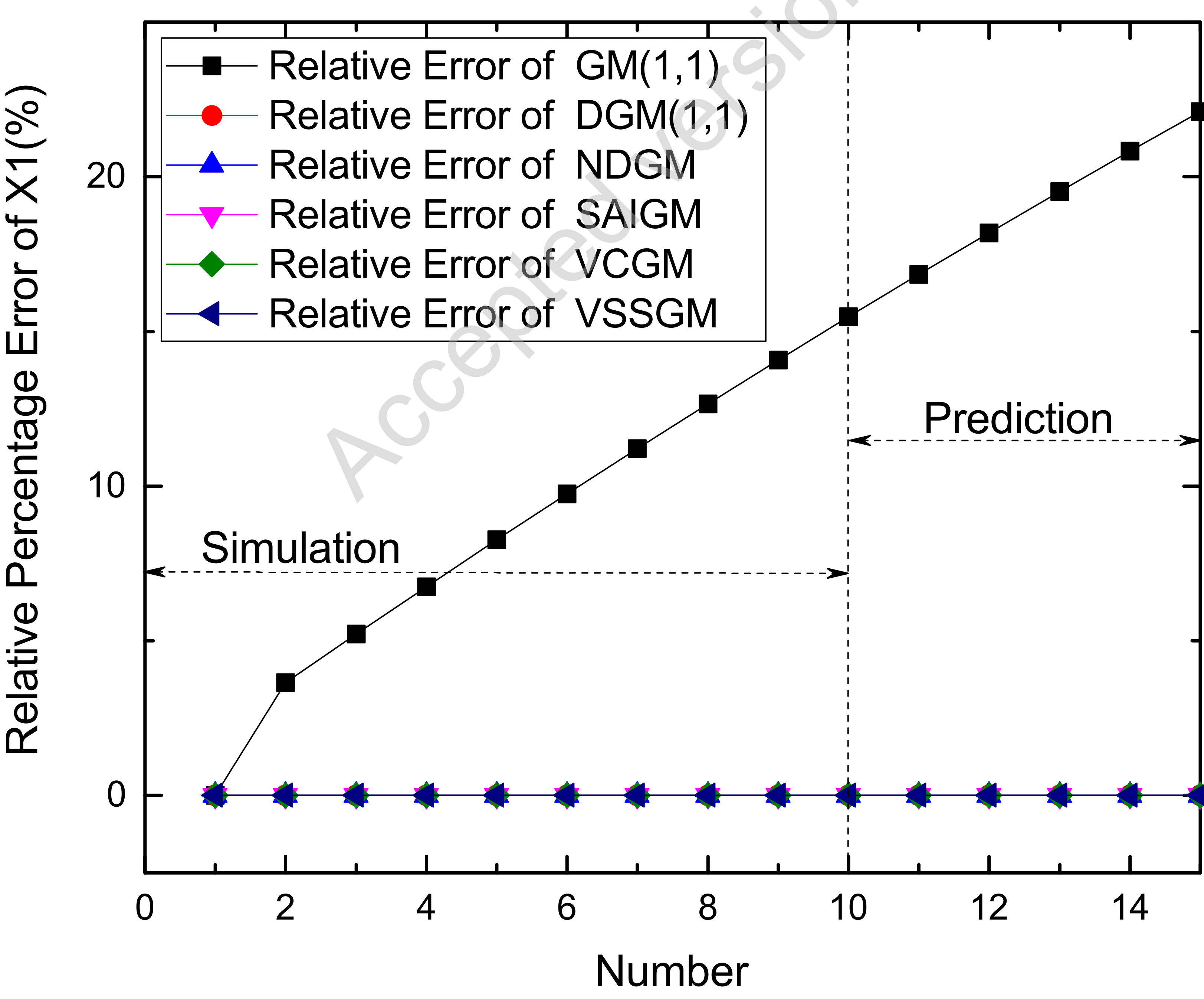
i =1, 2, 3, 4
$x^{(0)}(k) + a\, z^{(1)}(k) = f_i$
f_4
f_3
f_2
f_1
run through
$f_1 = b_4$
$f_2 = b_3k + b_4$
$f_3 = b_2k^2 + b_3k + b_4$
$f_4 = b_1k^3 + b_2k^2 + b_3k + b_4$
Optimization of structure
$\hat{x}^{(0)}(k) = a_1\, e^k$
$\hat{x}^{(0)}(k) = a_2\, e^k + d_2$
$\hat{x}^{(0)}(k) = a_3\, e^k + c_3\, k + d_3$
$\hat{x}^{(0)}(k) = a_4\, e^k + b_4\, k^2 + c_4\, k + d_4$
$x^{(0)}(k)$: Original sequence
Errors
Comparison & optimization

Figure

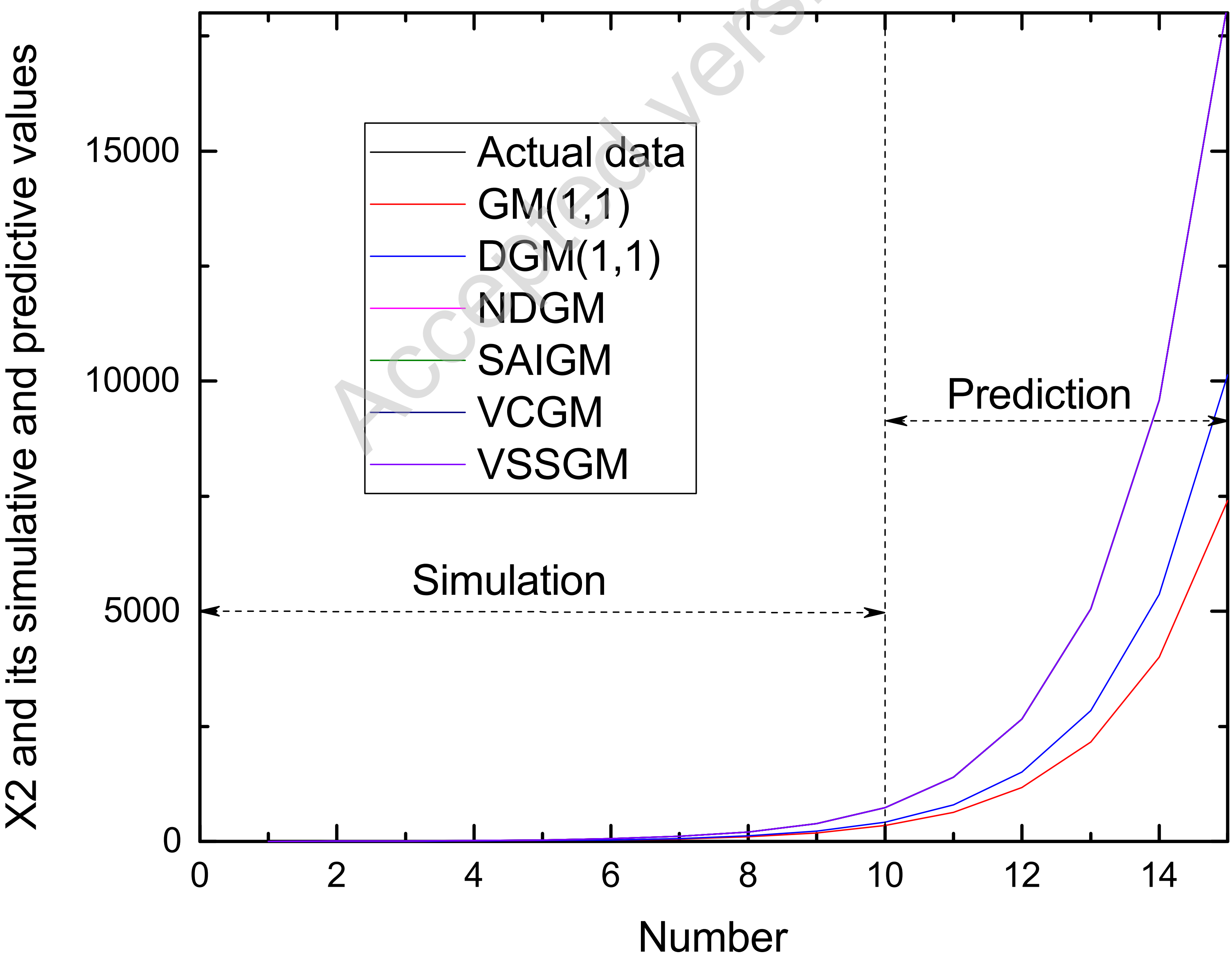


Actual data
GM(1,1)
DGM(1,1)
NDGM
SAIGM
VCGM
VSSGM
Prediction
Simulation
X1 and its simulative and predictive values
Number
0
1000
2000
3000
4000
5000
6000
0
2
4
6
8
10
12
14

Figure

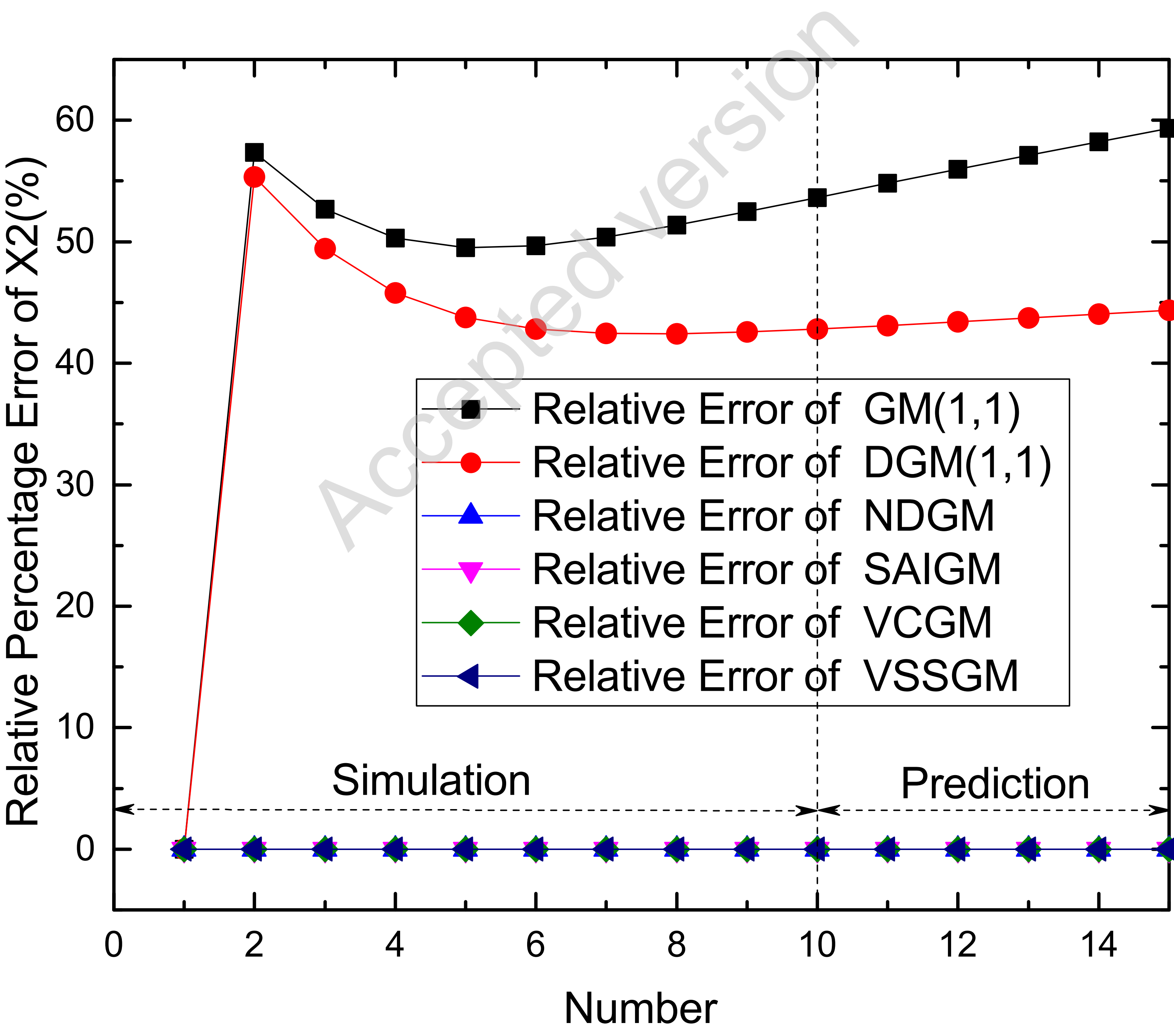

Relative Error of GM(1,1)
Relative Error of DGM(1,1)
Relative Error of NDGM
Relative Error of SAIGM
Relative Error of VCGM
Relative Error of VSSGM
Prediction
Simulation
Relative Percentage Error of X1(%)
Number
0
10
20
0
2
4
6
8
10
12
14

Figure



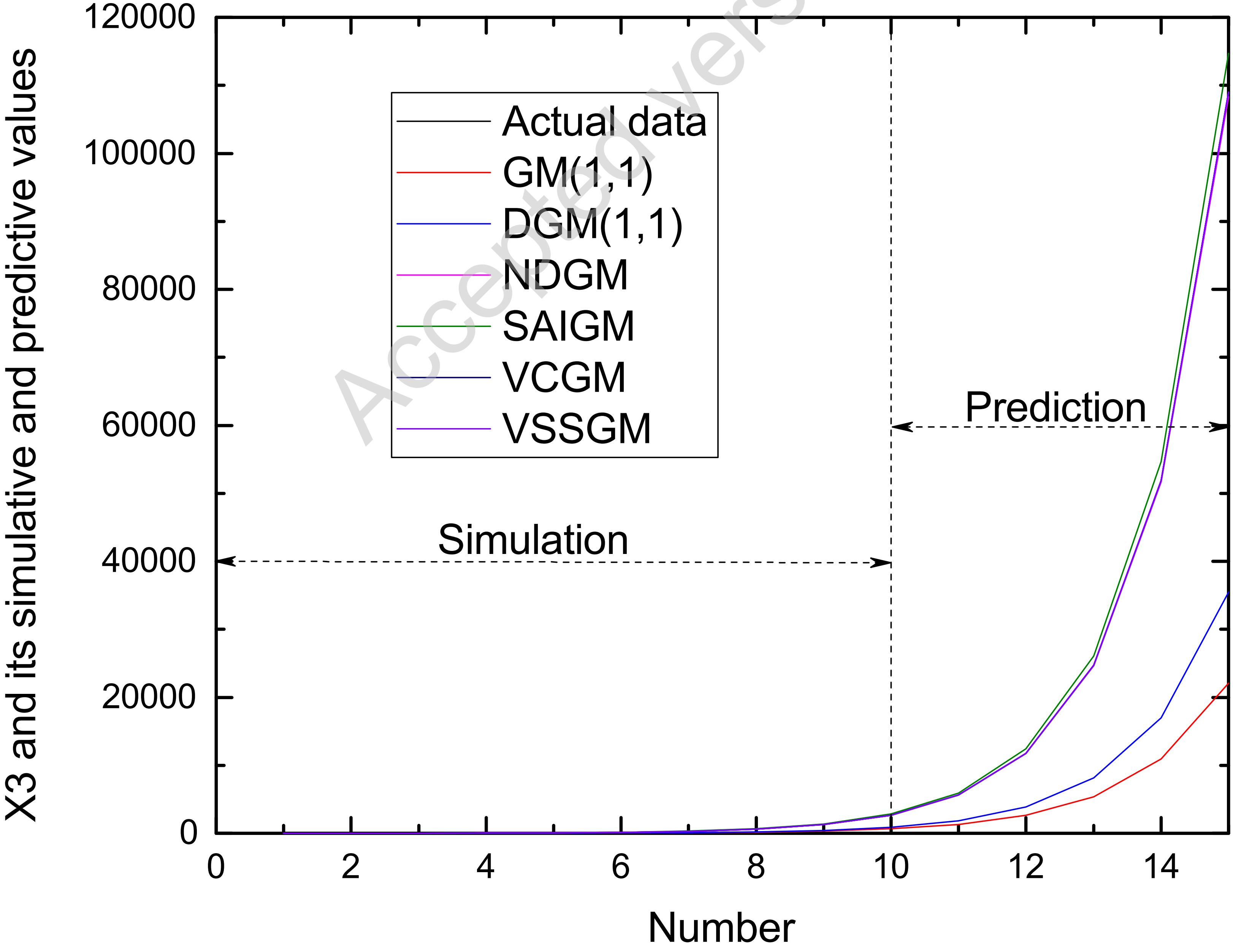
X2 and its simulative and predictive values
15000
10000
5000
0
0
2
4
6
8
10
12
14
Number
Actual data
GM(1,1)
DGM(1,1)
NDGM
SAIGM
VCGM
VSSGM
Simulation
Prediction

Figure



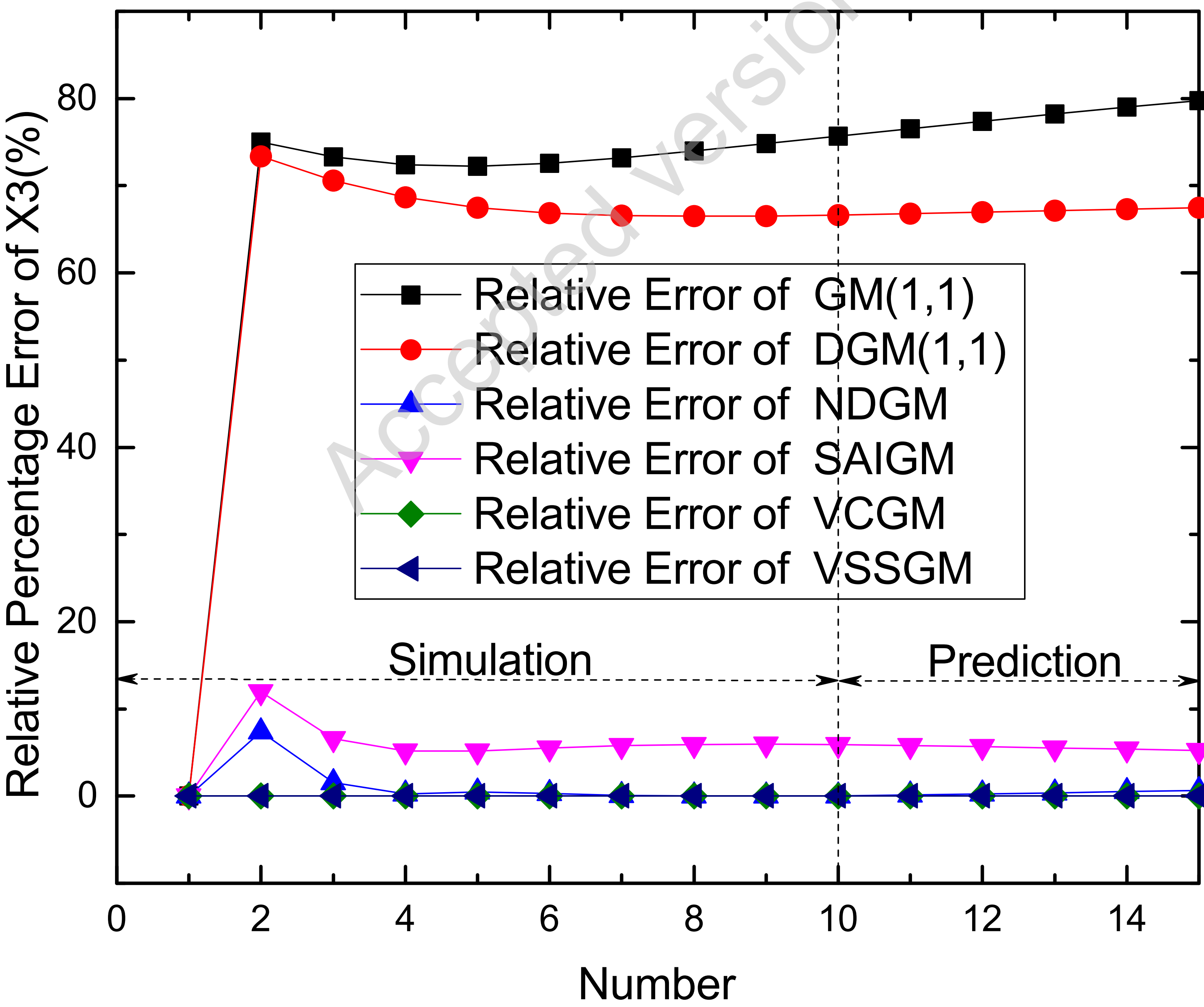
Relative Percentage Error of X2(%)
60
50
40
30
20
10
0
0
2
4
6
8
10
12
14
Number
Relative Error of GM(1,1)
Relative Error of DGM(1,1)
Relative Error of NDGM
Relative Error of SAIGM
Relative Error of VCGM
Relative Error of VSSGM
Simulation
Prediction

Figure



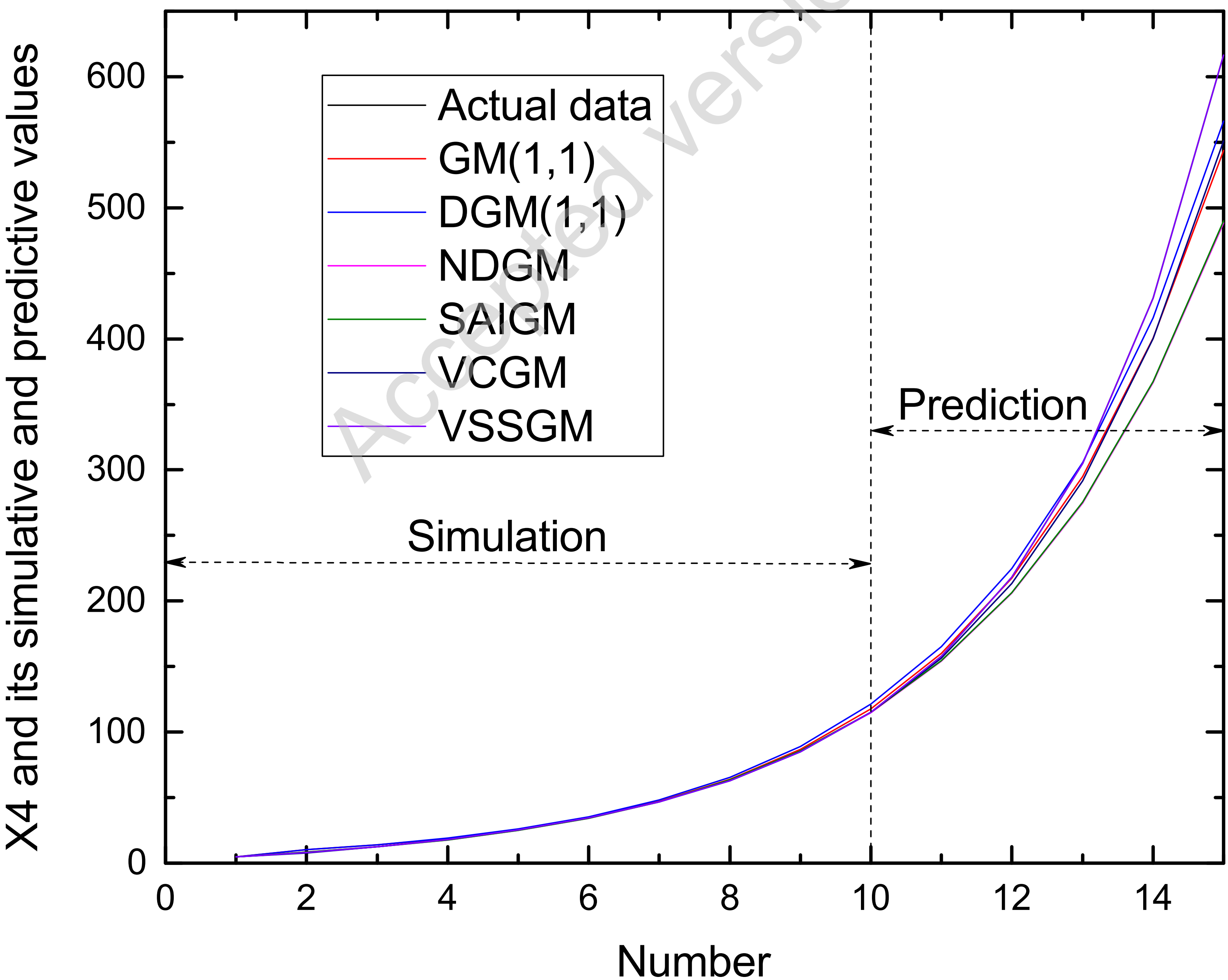
Actual data
GM(1,1)
DGM(1,1)
NDGM
SAIGM
VCGM
VSSGM
Prediction
Simulation
120000
100000
80000
60000
40000
20000
0
0
2
4
6
8
10
12
14
Number
X3 and its simulative and predictive values

Figure



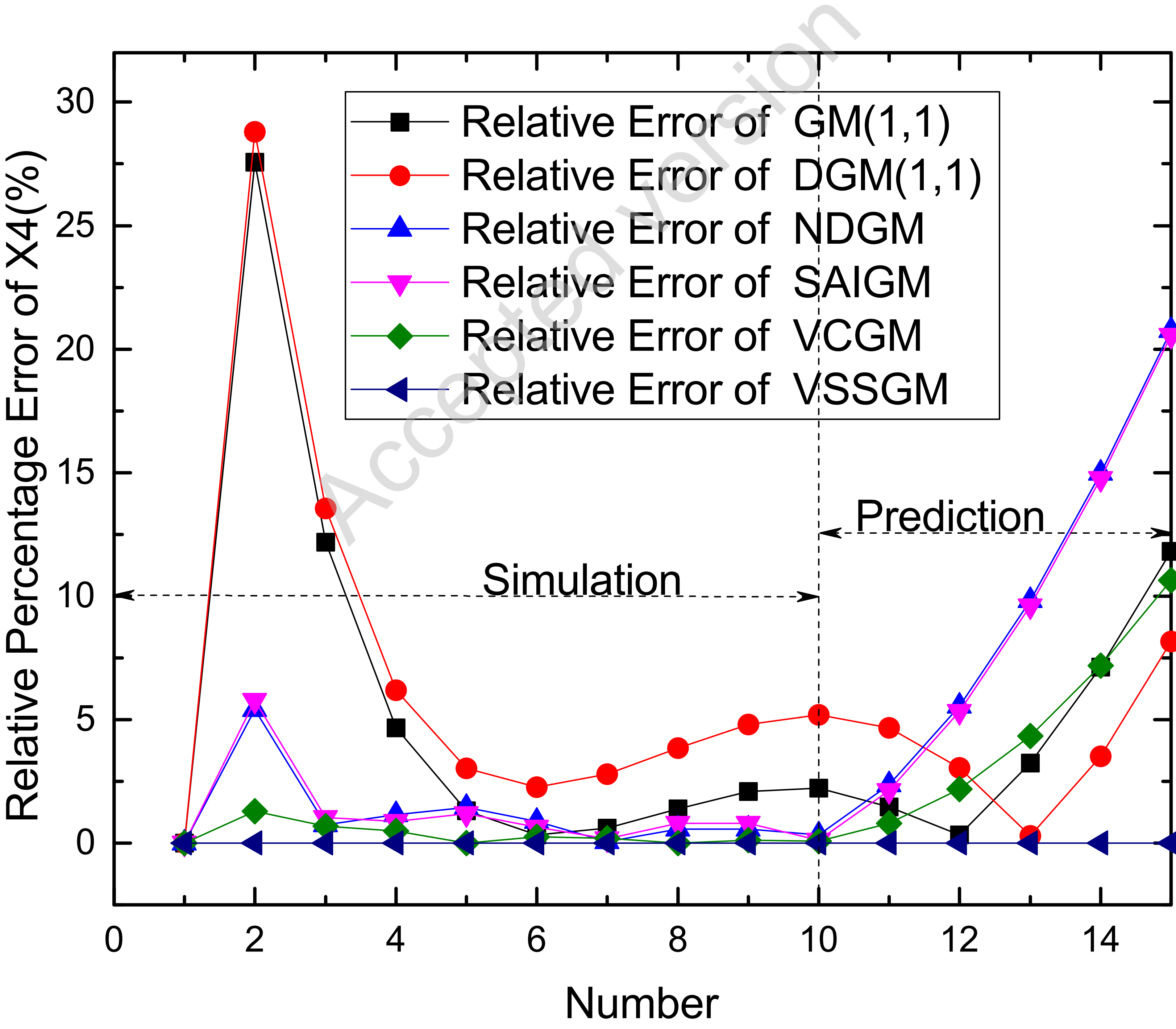
Relative Percentage Error of X3(%)
Number
Relative Error of GM(1,1)
Relative Error of DGM(1,1)
Relative Error of NDGM
Relative Error of SAIGM
Relative Error of VCGM
Relative Error of VSSGM
Simulation
Prediction
0
20
40
60
80
0
2
4
6
8
10
12
14

Figure



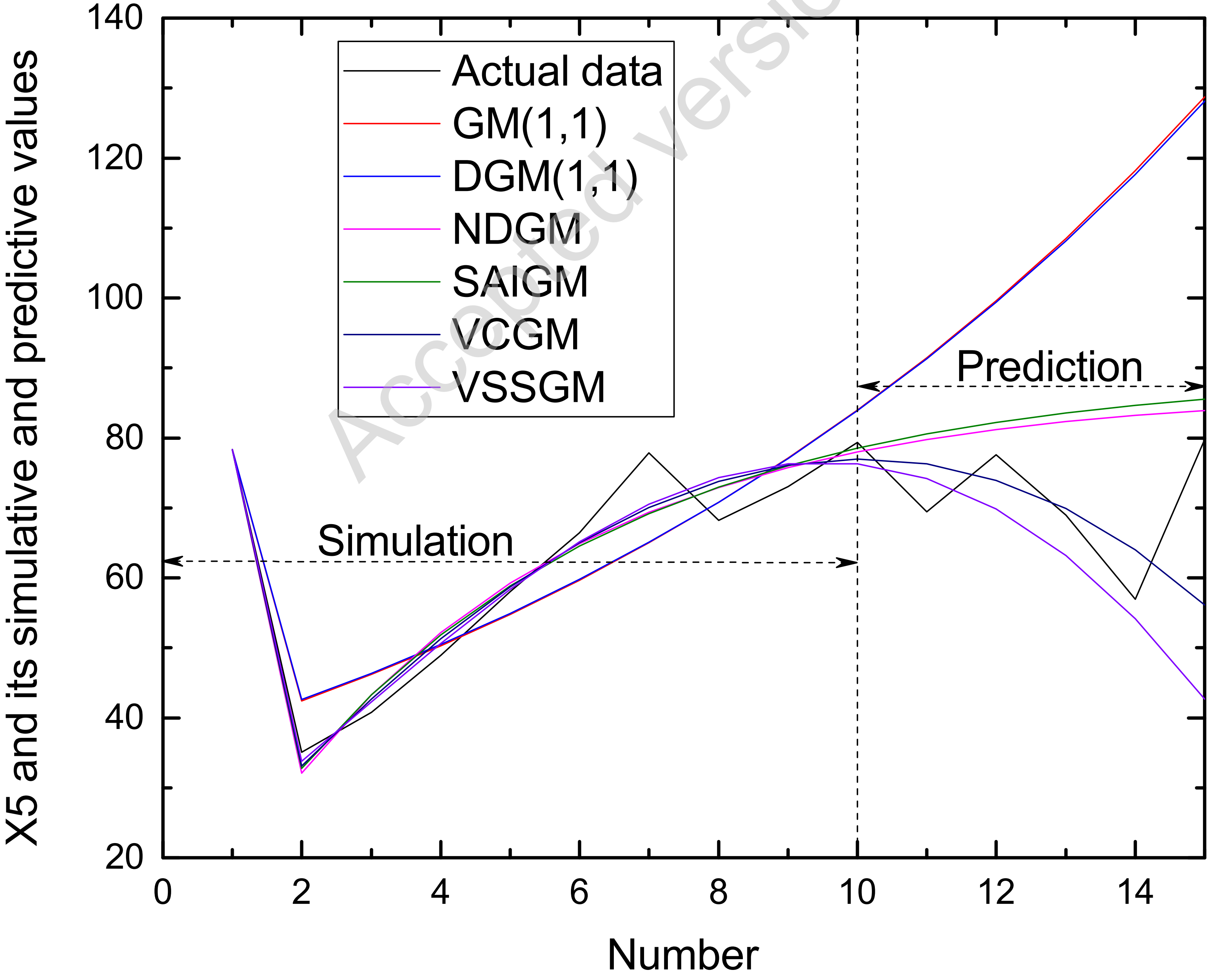
Actual data
GM(1,1)
DGM(1,1)
NDGM
SAIGM
VCGM
VSSGM
Prediction
Simulation
X4 and its simulative and predictive values
Number
0
100
200
300
400
500
600
0
2
4
6
8
10
12
14

Figure



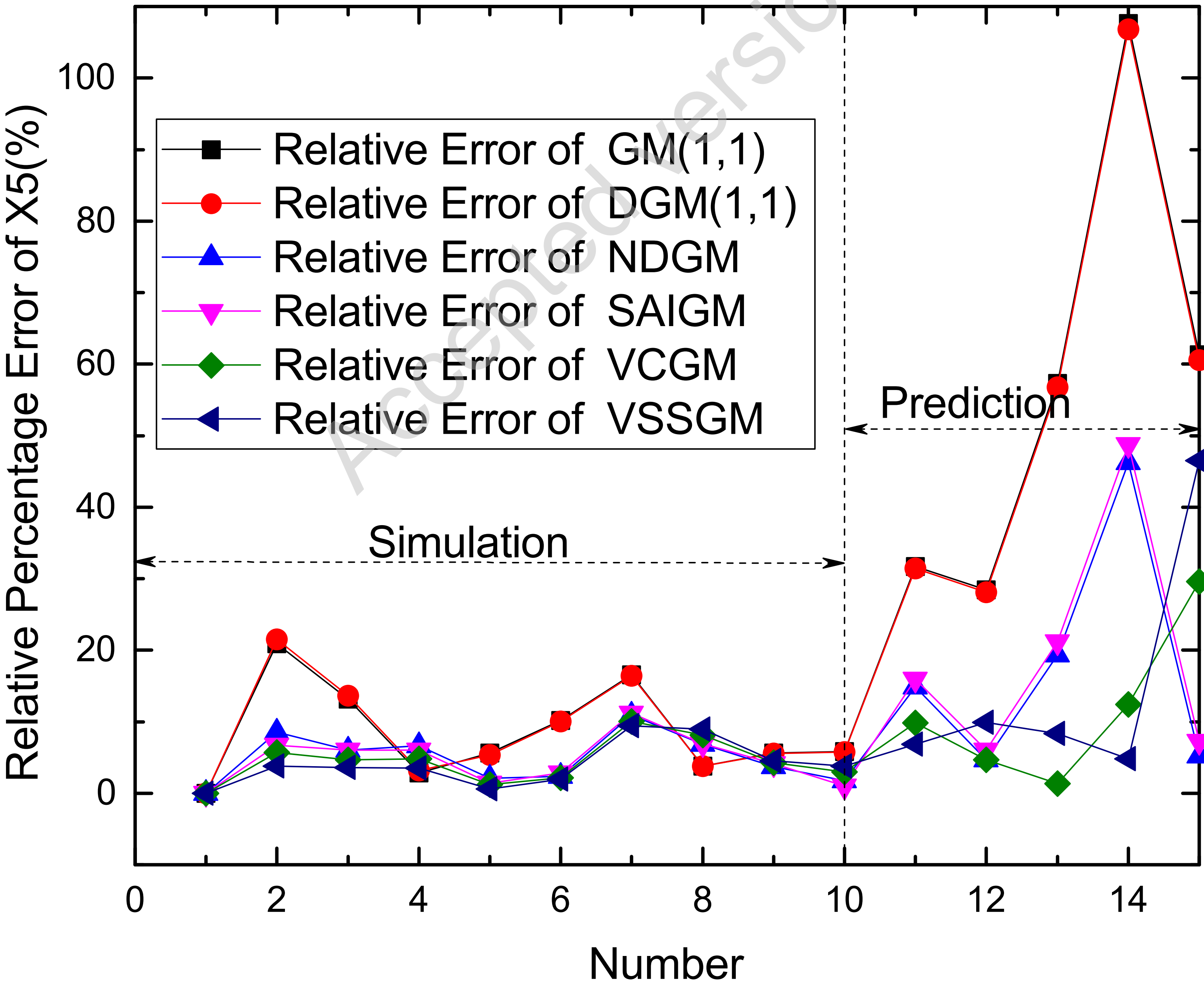
Relative Error of GM(1,1)
Relative Error of DGM(1,1)
Relative Error of NDGM
Relative Error of SAIGM
Relative Error of VCGM
Relative Error of VSSGM
Prediction
Simulation
Relative Percentage Error of X4(%)
Number
0
2
4
6
8
10
12
14
5
15
20
25
30

Figure

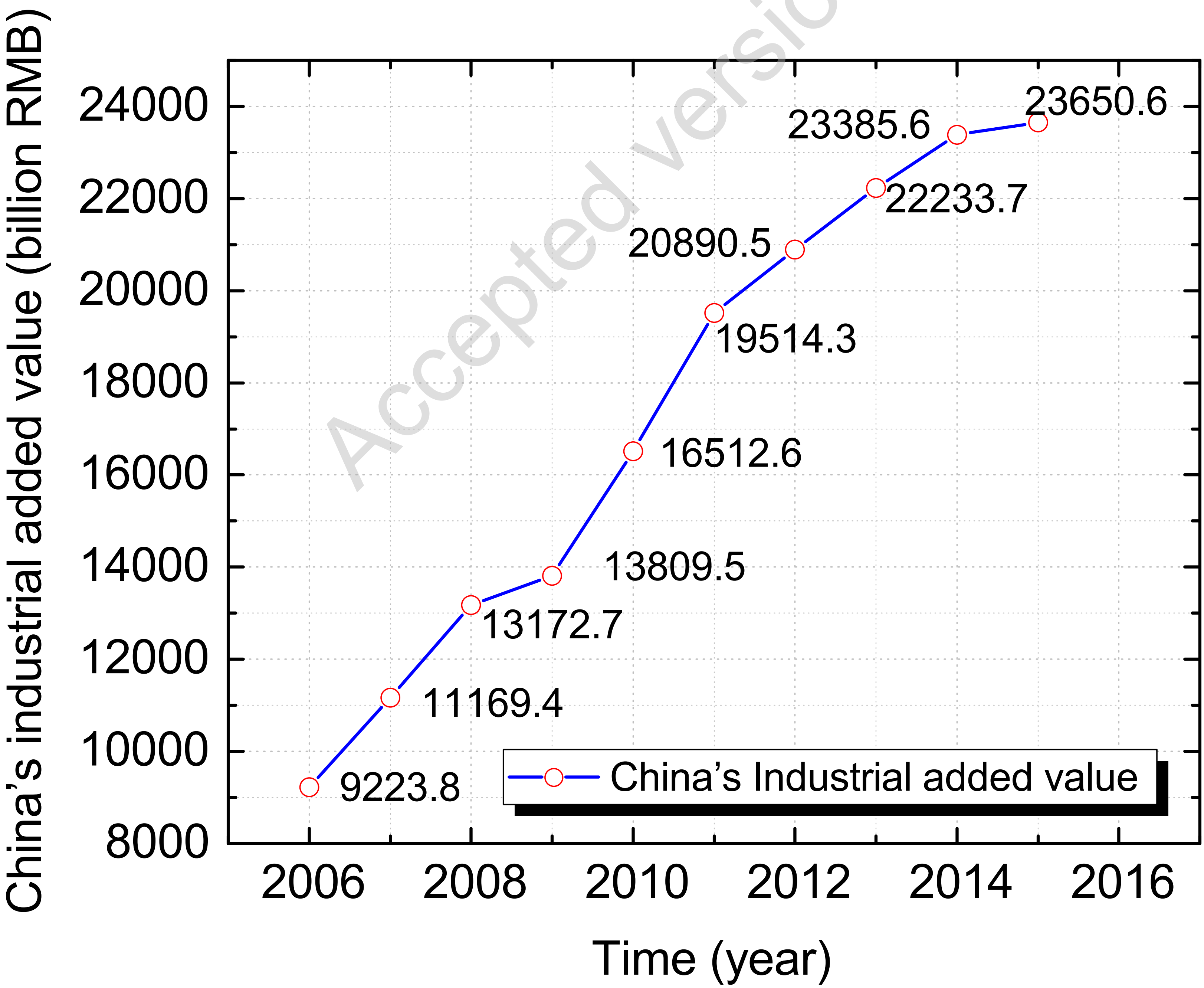


Actual data
GM(1,1)
DGM(1,1)
NDGM
SAIGM
VCGM
VSSGM
Simulation
Prediction
X5 and its simulative and predictive values
Number
140
120
100
80
60
40
20
0
2
4
6
8
10
12
14

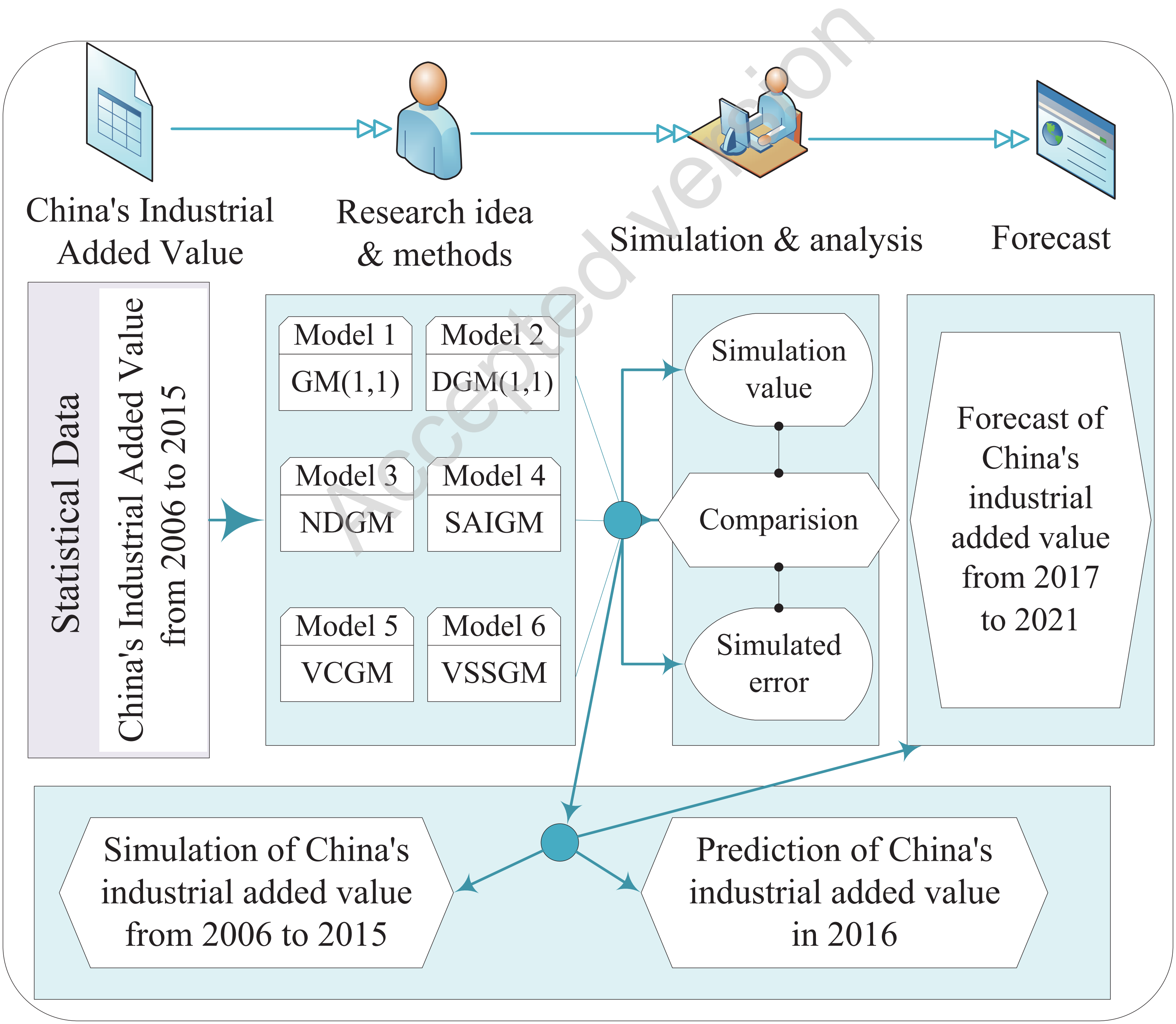
Figure


Relative Error of GM(1,1)
Relative Error of DGM(1,1)
Relative Error of NDGM
Relative Error of SAIGM
Relative Error of VCGM
Relative Error of VSSGM
Prediction
Simulation
Relative Percentage Error of X5(%)
Number
0
20
40
60
80
100
0
2
4
6
8
10
12
14

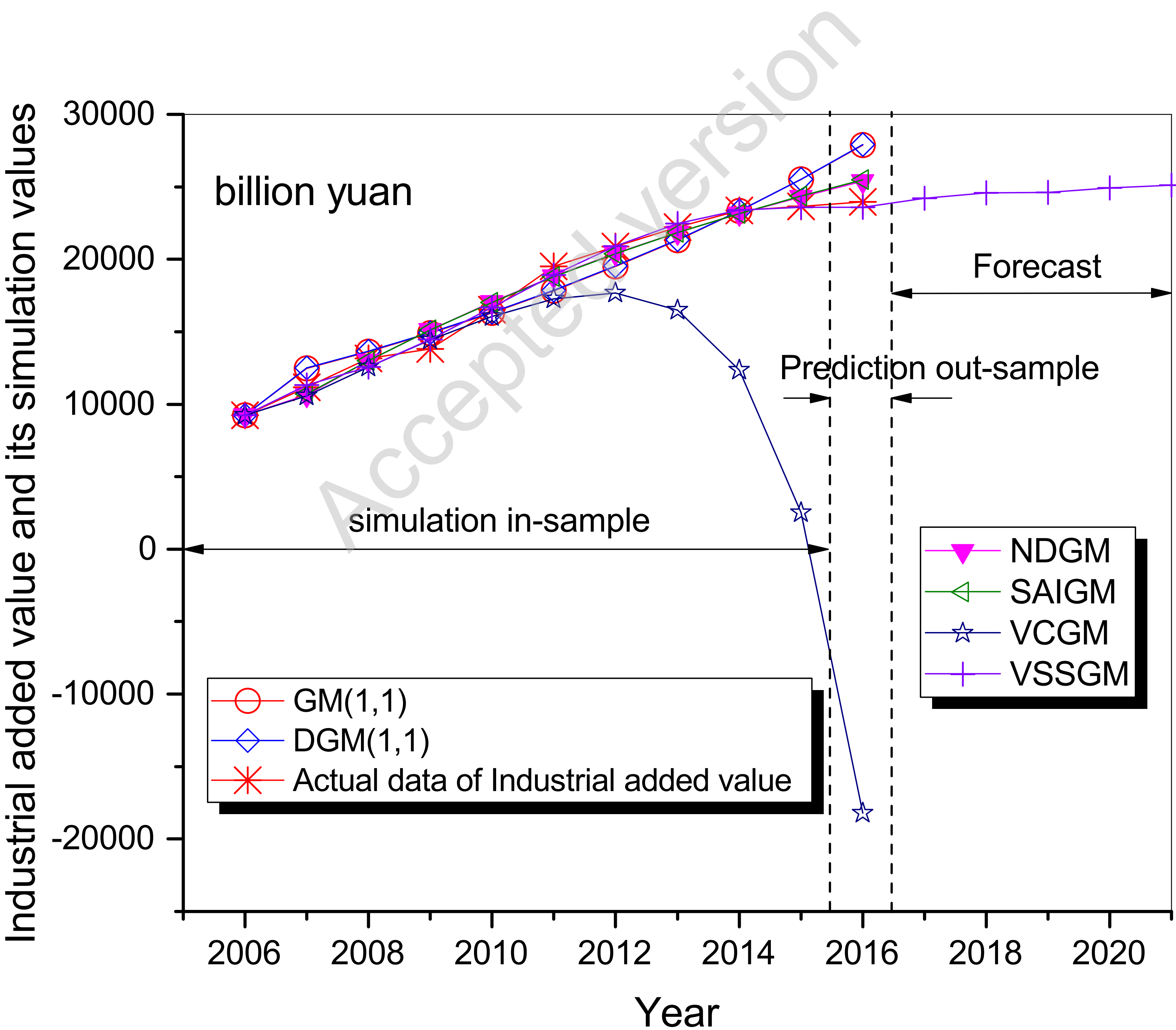
Figure


China's industrial added value (billion RMB)
24000
22000
20000
18000
16000
14000
12000
10000
8000
2006
2008
2010
2012
2014
2016
Time (year)
9223.8
11169.4
13172.7
13809.5
16512.6
19514.3
20890.5
22233.7
23385.6
23650.6
China's Industrial added value

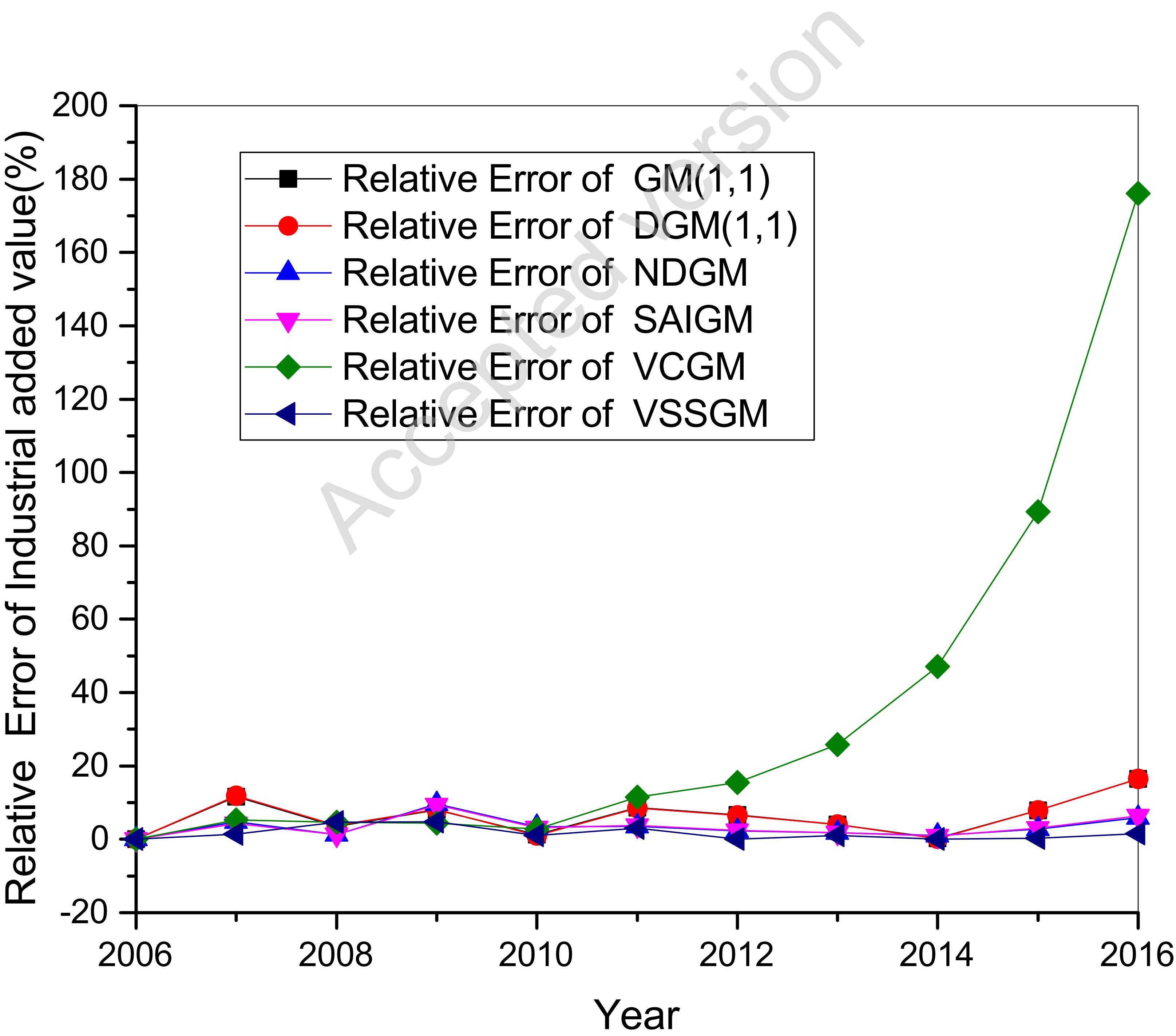
Figure


China's Industrial Added Value
Research idea & methods
Simulation & analysis
Forecast
Statistical Data
China's Industrial Added Value from 2006 to 2015
Model 1
GM(1,1)
Model 2
DGM(1,1)
Model 3
NDGM
Model 4
SAIGM
Model 5
VCGM
Model 6
VSSGM
Simulation value
Comparision
Simulated error
Forecast of China's industrial added value from 2017 to 2021
Simulation of China's industrial added value from 2006 to 2015
Prediction of China's industrial added value in 2016

Figure


billion yuan
Forecast
Prediction out-sample
simulation in-sample
NDGM
SAIGM
VCGM
VSSGM
GM(1,1)
DGM(1,1)
Actual data of Industrial added value
Industrial added value and its simulation values
Year
30000
20000
10000
0
-10000
-20000
2006
2008
2010
2012
2014
2016
2018
2020
Accepted version

Figure



Relative Error of GM(1,1)
Relative Error of DGM(1,1)
Relative Error of NDGM
Relative Error of SAIGM
Relative Error of VCGM
Relative Error of VSSGM
Relative Error of Industrial added value(%)
Year
200
180
160
140
120
100
80
60
40
20
0
-20
2006
2008
2010
2012
2014
2016

Highlights

(1) VSSGM model has characteristics of varistructure and speed adaption

(2) possess alterable structure and strong adaptability to perturbation sequence.

(3) simulate and forecast linear autoregressive sequence unbiasedly.

(4) proposed model has better performance than traditional GM(1,1) derived models.

(5) China's industrial added value has been predicted effectively.